On complex numbers I – Part 2

Roots of complex numbers, trig identities via complex numbers, the exponential form of a complex number, and properties of $|z|$ and $\mathrm{Arg}(z)$

By Chris Fenwick

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fenwick_chris@hotmail.com, uczlcfe@ucl.ac.uk

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1.13 On finding roots of a complex number

1.13.1 The general roots of a complex number

In section 1.7.4 we saw how to find the square root of a complex number z when z was expressed in Cartesian form. The approach used in that section could, in principle, be used to find cube roots, fourth roots, etc., but the algebra would become very complicated very quickly. DeMoivre's theorem allows us to find roots of complex numbers much more easily.

For example, given $z = -2 + 2i$ we can find the cube roots of z as follows: $r = |z| = \sqrt{8}$, and $\theta = \arg(z) = \tan^{-1}\big(2/(-2)\big) + \pi = 3\pi/4$. Therefore,

$$z^{1/3} = \left(\sqrt{8}\right)^{1/3}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^{1/3} = \left(\sqrt{8}\right)^{1/3}\left(\cos\frac{3\pi}{12} + i\sin\frac{3\pi}{12}\right). \qquad [*]$$

We can easily check this as

$$\left[\left(\sqrt{8}\right)^{1/3}\left(\cos\frac{3\pi}{12} + i\sin\frac{3\pi}{12}\right)\right]^3 = \sqrt{8}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -2 + 2i\,.$$

But we know that $\sqrt[3]{z}$ has three roots so what are the other two roots? Well, another solution is

$$z^{1/3} = \left(\sqrt{8}\right)^{1/3}(\cos 11\pi/12 + i\sin 11\pi/12)\,,$$

and yet another solution is

$$z^{1/3} = \left(\sqrt{8}\right)^{1/3}(\cos 19\pi/12 + i\sin 19\pi/12)\,,$$

both of which can be similarly checked to equal $-2 + 2i$.

So the question is, How do we obtain these other two solutions? To answer this question notice that to get the other two roots we have moved from $3\pi/12$ to $11\pi/12$ to $19\pi/12$, i.e. we have added $8\pi/12$ to the argument of each successive root. More precisely, we can calculate all the roots based on the argument $8k\pi/12 = 2k\pi/3$, where $k = 0, 1, 2$. This is equivalent to saying that our argument must be $(\pi/2 + 2k\pi)/3$. As such we see that we have included in our argument the periodic nature of the sin and cos functions, i.e. $\cos(\theta + 2k\pi) = \cos\theta$ and $\sin(\theta + 2k\pi) = \sin\theta$.

Therefore equation [*] is extended to include this aspect of periodicity:

$$\begin{aligned} z^{1/3} &= \left(\sqrt{8}\right)^{\frac{1}{3}}\left(\cos\left(\frac{3\pi}{4}+2k\pi\right)+i\sin\left(\frac{3\pi}{4}+2k\pi\right)\right)^{1/3}, \\ &= \left(\sqrt{8}\right)^{\frac{1}{3}}\left(\cos\frac{3\pi/4+2k\pi}{3}+i\sin\frac{3\pi/4+2k\pi}{3}\right). \end{aligned}$$

from which we obtain *all* three roots to $z=-2+2i$ by setting $k=0,1,2$, namely

1) $k=0$:

$$z_0=\left(\sqrt{8}\right)^{\frac{1}{3}}\left(\cos\left(\frac{3\pi}{12}\right)+i\sin\left(\frac{3\pi}{12}\right)\right),$$

   this being called the **principal root**;

2) $k=1$:

$$z_1=\left(\sqrt{8}\right)^{\frac{1}{3}}\left(\cos\left(\frac{11\pi}{12}\right)+i\sin\left(\frac{11\pi}{12}\right)\right)\ ;$$

3) $k=2$:

$$z_2=\cos\left(\frac{19\pi}{12}\right)+i\sin\left(\frac{19\pi}{12}\right)=\cos\left(\frac{5\pi}{12}\right)-i\sin\left(\frac{5\pi}{12}\right),$$

   where z_2 has been converted so that its argument lies in $-\pi<\theta\leq\pi$.

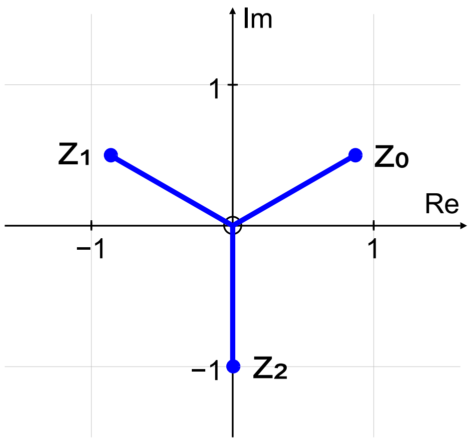
Our aim is therefore not just to perform the cube root of z but to do so in such a way that we obtain *all* roots which, when cubed, recover our original complex number. Formally speaking, we don't say that we want to find $z_k=z^{1/3}$ for a complex number z. We say that we want all solutions z_k such that $z=z_k^3$. This then guarantees us stating *all* roots z_k.

Notice that root z_2 can also be arrived at directly by setting $k=-1$ instead of $k=3$:

$$z_{-1}=\cos\left(\frac{3\pi/4-2\pi}{3}\right)+i\sin\left(\frac{3\pi/4-2\pi}{3}\right)=\cos\left(\frac{5\pi}{12}\right)-i\sin\left(\frac{5\pi}{12}\right).$$

Expressing the root in this way automatically keeps the argument in the required interval of $-\pi<\theta\leq\pi$.

Also notice that the roots are equiangular with respect to each other, changing from $3\pi/12$ to $11\pi/12$ to $19\pi/12$, so that each root is $2\pi/3$ away from the other two roots, as illustrated in the Argand diagram below:



We could have continued finding more roots by putting $k = 3, 4, 5, \ldots$ If $k = 3$ we have

$$z_3 = \cos\left(\frac{13\pi}{6}\right) + i\sin\left(\frac{13\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}i\,.$$

In other words we have cycled back to z_0. Using $k = 4$ would return us to z_1 and using $k = 5$ would return us to z_2 (confirm this). Similarly for $k = 6, 7, 8$ and $k = 9, 10, 11$, etc. So the cube roots of z cycle round every $k + 3$. The same is true if k is a negative integer. So there are in fact an infinite number of cube roots to $z = i$, but only three distinct roots.

Therefore, in general, given $z = r(\cos\theta + i\sin\theta)$, the n distinc n^{th} roots of $z = \cos\theta + i\sin\theta$ are given by

$$z^{1/n} = r^{1/n}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right) \qquad (42)$$

where $k = 0, \pm1, \pm2, \pm3, \ldots, \pm(n-1)$, and remembering that $-\pi < \theta \leq \pi$. The proof of (42**Error! Reference source not found.**), along with its more precise definition, is given in section 1.13.2 below.

There are several point to note about equation (42**Error! Reference source not found.**):

- it is the general expression for finding the first n roots of a complex number z;
- it includes the feature that the roots are equiangular;

- it has the feature that, although k has n values ranging from $k = 0$ to $k = \pm(n-1)$ we can in fact extend k beyond $\pm(n-1)$: z can in fact have an infinite number of n sets of roots, with the first set of n roots being $z_0, z_1, z_2, \ldots, z_{n-1}$, the second set of n roots being $z_n, z_{n+1}, z_{n+2}, \ldots, z_{2n-1}$, the thirds set of n roots being $z_{2n}, z_{2n+1}, z_{2n+2}, \ldots, z_{3n-1}$, etc.
- the first set of n roots are the distinct roots of z. All other roots are repetitions of the roots in this first set of n roots.
- if we choose to count k in positive integers then we will have to adjust the argument of certain roots in order to satisfy the interval of $-\pi < \theta \leq \pi$. Otherwise we need to use the relevant negative integer k values in order to automatially get the roots in this aforementioned interval.

<u>*Example 1:*</u> Consider finding the four 4th roots of $z = 1 + i$. In this case we have $r = |z| = \sqrt{2}$ and $\theta = \operatorname{Arg}(z) = \pi/4$. Hence $z = \sqrt{2}(\cos \pi/4 + i \sin \pi/4)$. Recasting this so as to take account of the periodic nature of sin and cos we have

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)\right).$$

for $k = 0, 1, 2, 3$. Now taking the fourth roots we have

$$\begin{aligned} \sqrt[4]{z} \quad = \quad z^{1/4} \quad &= \quad \left(\sqrt{2}\right)^{1/4}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)\right)^{1/4}, \\ &= \quad \left(\sqrt{2}\right)^{1/4}\left(\cos\left(\frac{\pi/4 + 2k\pi}{4}\right) + i\sin\left(\frac{\pi/4 + 2k\pi}{4}\right)\right). \end{aligned}$$

To find the principal fourth roots we set $k = 0$ to obtain

$$z_0 = \left(\sqrt{2}\right)^{1/4}\left(\cos\left(\frac{\pi}{16}\right) + i\sin\left(\frac{\pi}{16}\right)\right).$$

The second root is obtained by setting $k = 1$:

$$z_1 = \left(\sqrt{2}\right)^{1/4}\left(\cos\left(\frac{9\pi}{16}\right) + i\sin\left(\frac{9\pi}{19}\right)\right).$$

The third and fourth roots are obtained by setting $k = 2, 3$, to give $z_2 = \left(\sqrt{2}\right)^{1/4}(\cos(17\pi/16) + i\sin(17\pi/16))$, and $z_3 = \left(\sqrt{2}\right)^{1/4}(\cos(26\pi/16) + i\sin(26\pi/16))$. We now adjust the arguments of z_2 and z_3 so that they fall within the interval $-\pi < \theta \leq \pi$. Hence

$$z_2 = \left(\sqrt{2}\right)^{1/4}\left(\cos\left(-\frac{15\pi}{16}\right) + i\sin\left(-\frac{15\pi}{16}\right)\right) = \left(\sqrt{2}\right)^{1/4}\left(\cos\left(\frac{15\pi}{16}\right) - i\sin\left(\frac{15\pi}{16}\right)\right),$$

and

$$z_3 = \left(\sqrt{2}\right)^{1/4}\left(\cos\left(\frac{-7\pi}{16}\right) + i\sin\left(\frac{-7\pi}{16}\right)\right) = \left(\sqrt{2}\right)^{1/4}\left(\cos\left(\frac{7\pi}{16}\right) - i\sin\left(\frac{7\pi}{16}\right)\right).$$

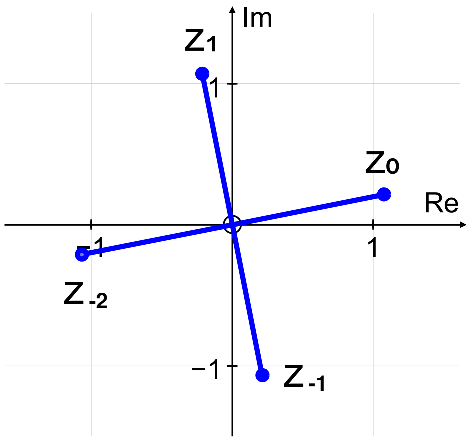
On the other hand we can set $k = -1$ and $k = -2$ to get the correct roots immediately:

$$z_{-1} = \left(\sqrt{2}\right)^{1/4}\left(\cos\left(\frac{-7\pi}{16}\right) + i\sin\left(\frac{-7\pi}{16}\right)\right) = \left(\sqrt{2}\right)^{1/4}\left(\cos\left(\frac{7\pi}{16}\right) - i\sin\left(\frac{7\pi}{16}\right)\right),$$

and

$$z_{-2} = \left(\sqrt{2}\right)^{1/4}\left(\cos\left(\frac{-15\pi}{16}\right) + i\sin\left(\frac{-15\pi}{16}\right)\right) = \left(\sqrt{2}\right)^{1/4}\left(\cos\left(\frac{15\pi}{16}\right) - i\sin\left(\frac{15\pi}{16}\right)\right).$$

Plotting these four roots on an Argand diagram we have



As for the previous example when we were finding the cube roots of $z = i$, so here also the roots are equally spaced out in terms of there angles with repsect to each other. In fact all roots are separated by $\pi/2$ radians, as can be seen by looking at the change in angle between all four roots.

<u>Example 2:</u> Consider finding the seven 7th roots of $z = -3 - 2i$. In this case we have $r = |z| = \sqrt{13}$ and $\theta = \tan(-2/(-3)) - \pi \approx -2.55$ radians. Hence $z = \sqrt{13}(\cos(-2.55) + i\sin(-2.55))$. Recasting this so as to take account of the periodic nature of sin and cos we have

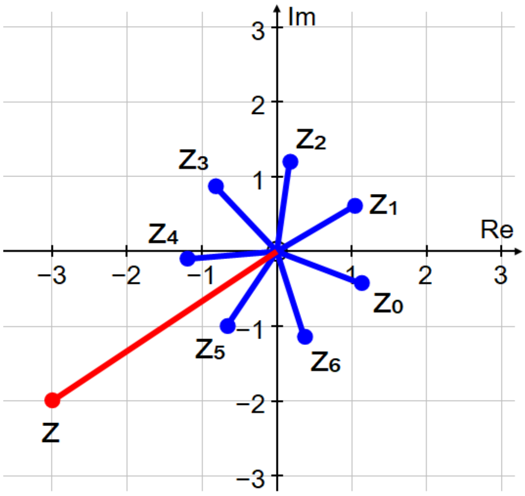
$$z = \sqrt{13}(\cos(-2.55 + 2k\pi) + i\sin(-2.55 + 2k\pi))\,.$$

for $k = 0, 1, 2, 3, 4, 5, 6$. Now taking the seventh roots we have

$$\begin{aligned}\sqrt[7]{z} = z^{1/7} &= \left(\sqrt{13}\right)^{1/7}(\cos(-2.55 + 2k\pi) + i\sin(-2.55 + 2k\pi))^{1/7}\,, \\ &= \left(\sqrt{13}\right)^{1/7}\left(\cos\left(\frac{-2.55 + 2k\pi}{7}\right) + i\sin\left(\frac{-2.55 + 2k\pi}{7}\right)\right).\end{aligned}$$

It is left as an exercise for you to find the seven individual seventh roots, all in principal argument form.

These seven roots are illustrated below:

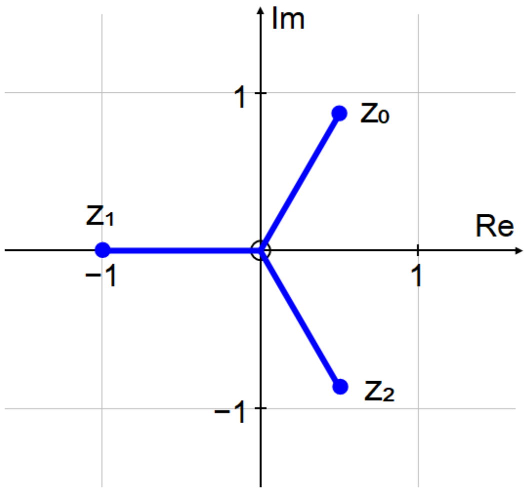


Example 3: Knowing that the roots of complex number z are equiangular we can find the location of all roots of z given one of the roots of a z. For example, if we know that one cube root of a complex number z is $z_1 = -1$, what are the other two roots? Well, we know that the sum of the three arguments must be 2π.

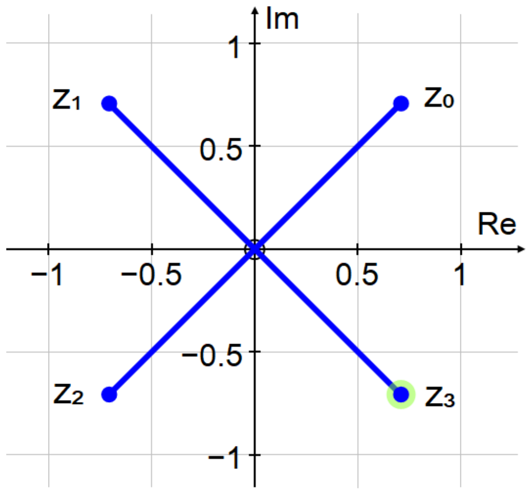
If we let ϕ be the angle between each root we have $3\phi = 2\pi$, implying that $\phi = 2\pi/3$. Since $\mathrm{Arg}(z_1) = \pi$ we have $\arg(z_0) = \pi - 2\pi/3$, hence $\mathrm{Arg}(z_0) = \arg(z_0) = \pi/3$. Also, $\arg(z_2) = \pi + 2\pi/3 = 5\pi/3$, hence $\mathrm{Arg}(z_2) = -\pi/3$ (note that we could simply have added $2\pi/3$ to $\mathrm{Arg}(z_1)$ to get the next root, and again added $2\pi/3$ to this root to get the final root, all th ewhile readjusting our argument so as to be a principal argument).

Similarly, if we know that one fourth root of a complex number z is $z_2 = -1 - i$ we know that $4\phi = 2\pi$, again where ϕ be the angle between each root. Hence $\phi = \pi/2$ is the angle between each root. Now, $\mathrm{Arg}(z_2) = -3\pi/4$, therefore we simply need to add (or subtract) $\pi/2$ again and again until we obtain the other three roots. Adding gives us $-\pi/4$, $\pi/4$, $3\pi/4$. So the principal arguments of the roots are $\mathrm{Arg}(z_0) = \pi/4$, $\mathrm{Arg}(z_1) = \pi/4$, $\mathrm{Arg}(z_2) = -3\pi/4$, $\mathrm{Arg}(z_3) = -\pi/4$.

These two examples are illustrated below.



Given one root at $z_1 = -1$



Given one root at $z_2 = -1 - i$

<u>*Example 4:*</u> Notice one difference between the roots of $z = i$ and the roots of $z = 1 + i$. The former complex number is pure imaginary, and one of its roots is real. The latter complex number is neither pure imaginary nor pure real, and none of its roots are real. In general it is always the case that

i) complex number which are pure imaginary or pure real always have at least one real root;

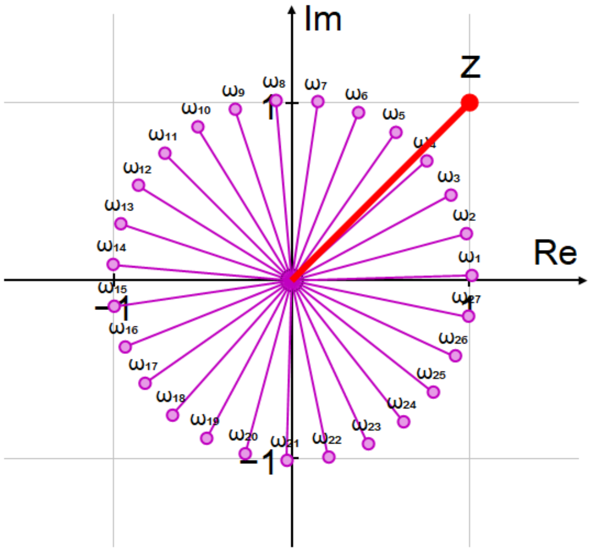
ii) complex complex which are neither pure real nor pure imaginary does not have any real roots, provided we are dealing with a finite number of roots. If we take the n^{th} root of z, and let $n \to \infty$, then one root of z will be real, this being $z_{k\to\infty} = 1$. To see this consider

$$z^{1/n} = r^{1/n}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right).$$

As $n \to \infty$, $1/n \to 0$. Therefore $r^{1/n} \to 1$ and $(\theta + 2k\pi)/n \to 0$. Hence

$$z_{k\to\infty} = \lim_{n\to\infty} z^{1/n} = 1(\cos 0 + i\sin 0) = 1\,.$$

This case can be illustrated as follows: for the complex number $z = 1 + i$ we can track the distribution of roots of $z^{1/n}$ from $n = 2$ onwards to see that none of them are ever real except at $\lim_{n\to\infty} z^{1/n}$. The diagram below shows the case of the twenty-seven 27^{th} roots of z (labelled as ω in the diagram), with $\text{Arg}(\omega_1) = 0.0291$ radians.



Example 5: When finding roots of complex numbers we note that substituting $k = 0$ into equation (42) gives us the principal value. This value is the principal value because of interval $-\pi < \theta \leq \pi$. Suppose, instead, that our interval for θ were to be $\pi \leq \theta < 3\pi$. what then would be our principal valur for the cube root of $z = i$?

Well, for $z = i$ we have

$$z^{1/3} = \cos\left(\frac{\pi/2 + 2k\pi}{3}\right) + i\sin\left(\frac{\pi/2 + 2k\pi}{3}\right).$$

For the interval $\pi \leq \theta < 3\pi$ our principal value will occur when $k = 3$, which will give us the first argument greater than, or equal to, π. Hence

$$z_3 = \cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right).$$

Similarly if we chose the interval $-3\pi < \theta \leq -\pi$ then our principal value would be the one including the first argument less than, or equal to, $-\pi$. This occurs when $k = -2$. Hence

$$z_{-2} = \cos\left(\frac{-7\pi}{6}\right) + i\sin\left(\frac{-7\pi}{6}\right).$$

Example 6: To find all values of $z = \sqrt[5]{-32}$ we proceed as follows: Finding r and θ in the usual way we have

$$-32 = 32\{\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)\}.$$

Applying DeMoivre's theorem we obtain

$$(-32)^{1/5} = (32)^{1/5}\{\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)\}^{1/5} = 2\left\{\cos\left(\frac{\pi + 2k\pi}{5}\right) + i\sin\left(\frac{\pi + 2k\pi}{5}\right)\right\}.$$

When $k = 0$, $\quad z_0 = 2(\cos\pi/5 \ + i\sin\pi/5)$,

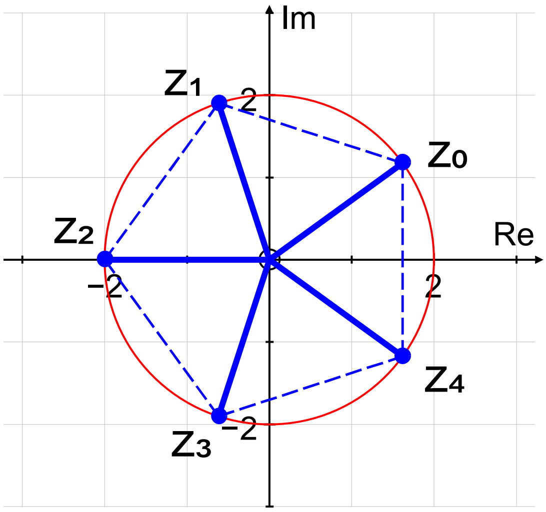
When $k = 1$, $\quad z_1 = 2(\cos 3\pi/5 \ + i\sin 3\pi/5)$,

When $k = 2$, $\quad z_2 = 2(\cos 5\pi/5 \ + i\sin 5\pi/5) = -2$,

When $k = 3$, $\quad z_3 = 2(\cos 7\pi/5 \ + i\sin 7\pi/5) = 2(\cos 3\pi/5 \ - i\sin 3\pi/5)$,

When $k = 4$, $\quad z_4 = 2(\cos 9\pi/5 \ + i\sin 9\pi/5) = 2(\cos\pi/5 \ - i\sin\pi/5)$.

Continuing to count $k = 5, 6, 7, \ldots$ or $k = -1, -2, -3, \ldots$ would only repeat the values of the roots above. Plotting these roots on an Argand diagram is illustrated below. Notice that all five roots lie on a circle of radius 2, and each complex number can be seen as the vertex of a pentagon.



Example 7: To find all values of $z = \left(-1 - i\sqrt{3}\right)^{1/4}$ we proceed as follows: Finding r and θ in the usual way we have two choices about how to express $-1 - i\sqrt{3}$ in polar form. We can either write

$$-1 - i\sqrt{3} = 2\left(\cos\left(\frac{4\pi}{3} + 2k\pi\right) + i\sin\left(\frac{4\pi}{3} + 2k\pi\right)\right),$$

and wait until we have found all four roots before adjusting our arguments to be in the interval $-\pi < \theta \leq \pi$, or we can express $-1 - i\sqrt{3}$ immediately in terms of its principal argument, i.e

$$-1 - i\sqrt{3} = 2(\cos(-2\pi/3 + 2k\pi) + i\sin(-2\pi/3 + 2k\pi)) .$$

In either case it does not make a difference. Provided the arguments of our roots are ultimately expressed in $-\pi < \theta \leq \pi$ we will always obtain the same values for our roots.

Hence applying DeMoivre's theorem on the latter equation we obtain

$$(-1-i\sqrt{3})^{1/4} = 2^{1/4}\left\{\cos\left(-\frac{2\pi}{3}+2k\pi\right)+i\sin\left(-\frac{2\pi}{3}+2k\pi\right)\right\}^{1/4}$$

$$= 2^{1/4}\left\{\cos\left(\frac{-2\pi/3+2k\pi}{4}\right)+i\sin\left(\frac{-2\pi/3+2k\pi}{4}\right)\right\}.$$

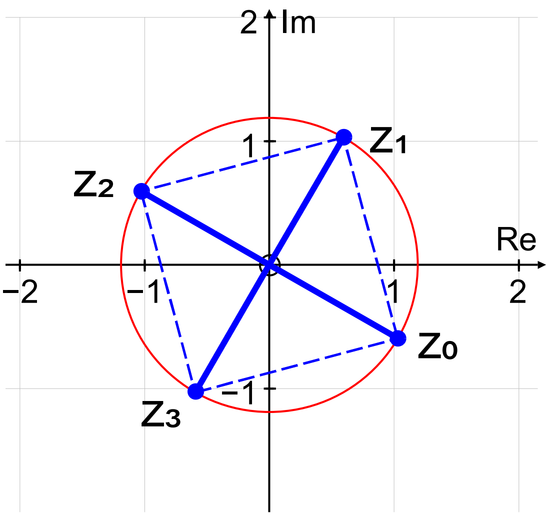
When $k = 0$, $\qquad z_0 = 2^{1/4}(\cos(-2\pi/12) + i\sin(-2\pi/12))$,

When $k = 1$, $\qquad z_1 = 2^{1/4}(\cos 4\pi/12 + i\sin 4\pi/12)$,

When $k = 2$, $\qquad z_2 = 2^{1/4}(\cos 10\pi/12 + i\sin 10\pi/12)$,

When $k = 3$, $\qquad z_3 = 2^{1/4}(\cos 16\pi/12 + i\sin 16\pi/12) = 2^{1/4}(\cos 8\pi/12 - i\sin 8\pi/12)$,

Plotting these roots on an Argand diagram is illustrated below. Notice that all four roots lie on a circle of radius $2^{1/4}$, and each complex number can be seen as the vertex of a square.



Example 8: To find all values of $z = [16i/(1+i)]^{1/8}$ we proceed as follows: Finding r and θ in the usual way we will separately express $16i$ and $1+i$ in polar form, then simplify the division via DeMoivre's theorem. Hence

$$z_1 = 16i \quad \Longrightarrow \quad z = 16\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right),$$

$$z_2 = 1+i \quad \Longrightarrow \quad \sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right).$$

There is no need to take account of the periodic nature of sin and cos at this stage since we are now going to divide z_1 by z_2 to form the single complex number z.

Hence

$$\frac{z_1}{z_2} = \frac{16i}{1+i} \quad = \quad \frac{16}{\sqrt{2}}.\frac{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}}{\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}},$$

$$= \quad \sqrt{128}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),$$

$$= \quad \sqrt{128}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)\right),$$

for $k = 0, 1, 2, 3 \ldots$ Taking the eighth roots we obtain

$$z^{1/8} \quad = \quad \left(\sqrt{128}\right)^{1/8}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)\right)^{1/8},$$

$$= \quad \left(\sqrt{128}\right)^{1/8}\left(\cos\left(\frac{\pi/4 + 2k\pi}{8}\right) + i\sin\left(\frac{\pi/4 + 2k\pi}{8}\right)\right).$$

When $k = 0$, $\quad z_0 = \left(\sqrt{128}\right)^{1/8}(\cos(\pi/32) + i\sin(\pi/32))$,

When $k = 1$, $\quad z_1 = \left(\sqrt{128}\right)^{1/8}(\cos 9\pi/32 + i\sin 9\pi/32)$,

When $k = 2$, $\quad z_2 = \left(\sqrt{128}\right)^{1/8}(\cos 17\pi/32 + i\sin 17\pi/32)$,

When $k = 3$, $\quad z_3 = \left(\sqrt{128}\right)^{1/8}(\cos 25\pi/32 + i\sin 25\pi/32)$,

When $k = 4$, $\quad z_3 = \left(\sqrt{128}\right)^{1/8}(\cos 33\pi/32 + i\sin 33\pi/32)$,

$$= \left(\sqrt{128}\right)^{1/8}(\cos 31\pi/32 - i\sin 31\pi/32),$$

When $k = 5$, $\quad z_3 = \left(\sqrt{128}\right)^{1/8}(\cos 41\pi/32 + i\sin 41\pi/32)$

$$= \left(\sqrt{128}\right)^{1/8}(\cos 23\pi/32 - i\sin 23\pi/32),$$

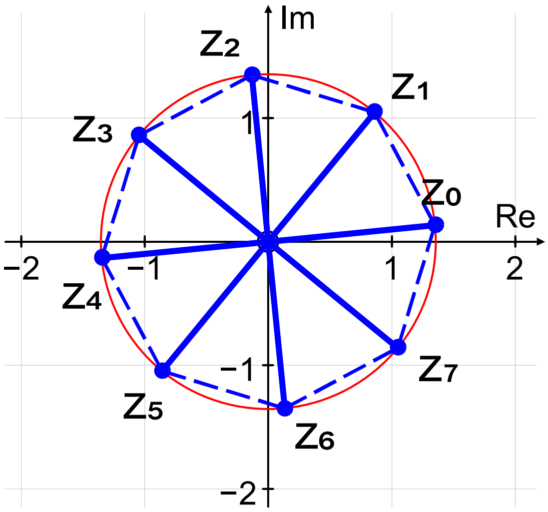
When $k = 6$, $\quad z_3 = \left(\sqrt{128}\right)^{1/8}(\cos 49\pi/32 + i\sin 49\pi/32)$

$$= \left(\sqrt{128}\right)^{1/8}(\cos 15\pi/32 - i\sin 15/32),$$

When $k = 7$, $\quad z_3 = \left(\sqrt{128}\right)^{1/8}(\cos 57\pi/32 + i\sin 57\pi/32)$

$$= \left(\sqrt{128}\right)^{1/8}(\cos 7\pi/32 - i\sin 7\pi/32).$$

Plotting these roots on an Argand diagram is illustrated below. Notice that all eight roots lie on a circle of radius $\left(\sqrt{128}\right)^{1/8}$, and each complex number can be seen as the vertex of an octogon.



Example 9: To find the exact values of the sum and product of the three roots of $z = (\cos 3\pi + i \sin 3\pi\ )^{1/3}$ we apply DeMoivre's theorem to obtain

$$z = \cos\left(\frac{3\pi + 2k\pi}{3}\right) + i \sin\left(\frac{3\pi + 2k\pi}{3}\right),$$

from which we have

for $k = 0,$ $\quad z_0 = \cos(\pi)\ + i \sin(\pi) = -1\,,$

for $k = 1,$ $\quad z_1 = \cos 5\pi/3\ + i \sin 5\pi/3 = \cos \pi/3\ - i \sin \pi/3\,,$

for $k = 2,$ $\quad z_2 = \cos 7\pi/3\ + i \sin 7\pi/3 = \ \cos \pi/3\ + i \sin \pi/3\,.$

Hence the sum of the roots is $z_1 + z_2 = -1 + 2\cos \pi/3$, and since $\cos(-\pi/3) = \cos(\pi/3)$ the product is $z_0 z_1 z_2 = -1(\cos \pi/3\ + i \sin \pi/3)(\cos(-\pi/3)\ + i \sin(-\pi/3)) = -2\cos(2\pi/3)$.

Exercise: Find the exact values of the sum and product of the five roots of $z = (\cos \pi + i \sin \pi\ )^{2/5}$.

Example 10: Given that $z = \cos\theta + i \sin\theta$ suppose we want to find $\sqrt{(1+z)/(1-z)}$ in the interval $0 < \theta < \pi$ and $\pi < \theta < 2\pi$. Firstly, for simplicty, let $\cos\theta + i \sin\theta \equiv c + is$. Then

$$\frac{1+z}{1-z} = \frac{1+c+is}{1-c-is} \qquad (*)$$

Now multiple top and bottom of (*) by the conjugate of the denominator of (*)

$$\frac{1+z}{1-z} \quad = \quad \frac{1+c+is}{1-c-is}.\frac{1-c+is}{1-c+is},$$

$$= \quad i.\frac{s}{1-c},$$

By using the double-angle trig identities we have $\cos\theta = 2\cos(\theta/2) = 1 - 2\sin^2(\theta/2)$ and $\sin\theta = \sin 2(\theta/2) = 2\sin(\theta/2).\cos(\theta/2)$. Hence

$$\frac{1+z}{1-z} = i.\frac{2\sin(\theta/2)\cos(\theta/2)}{1-(1-2\sin^2(\theta/2))},$$

which simplifies to

$$\frac{1+z}{1-z} = i\cot\left(\frac{\theta}{2}\right).$$

Therefore

$$\sqrt{\frac{1+z}{1-z}} = \left(i\cot\left(\frac{\theta}{2}\right)\right)^{1/2} = (i)^{1/2}.\sqrt{\cot\left(\frac{\theta}{2}\right)}\ .$$

We now need to find the two roots of $i^{1/2}$ which, by the usual method, are

$$\cos\frac{\pi}{4} + i.\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, \qquad \text{for } 0 < \theta < \pi$$

and

$$\cos\frac{5\pi}{4} + i.\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}. \qquad \text{for } \pi < \theta < 2\pi\ .$$

Hence, for $0 < \theta < \pi$

$$\sqrt{\frac{1+z}{1-z}} = \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\sqrt{\cot\left(\frac{\theta}{2}\right)} = (1+i)\sqrt{\frac{1}{2}\cot\left(\frac{\theta}{2}\right)}\ ,$$

and for $\pi < \theta < 2\pi$

$$\sqrt{\frac{1+z}{1-z}} = \left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\sqrt{\cot\left(\frac{\theta}{2}\right)} = (1+i)\sqrt{-\frac{1}{2}\cot\left(\frac{\theta}{2}\right)}\ .$$

Example 11: Now that we know about DeMoivre's theorem we can now solve polynomials in z. for eample, to solve $z^6 - z^3(1+i) + i = 0$ we can factorise this to be $(z^3-1)(z^3-i) = 0$.

Hence either $z^3 - 1 = 0$ and/or $z^3 - i = 0$, form which we get the six solutions, via DeMoivre's theorem, to be

- for $z^3 - 1 = 0$: $z_k = \cos(2k\pi/3) + i\sin(2k\pi/3)$ for $k = 0, 1, 2$;
- for $z^3 - i = 0$: $z_k = \cos\{(\pi + 2k\pi)/3\} + i\sin\{(\pi + 2k\pi)/3\}$ for $k = 0, 1, 2$.

Exercises: Solve the following polynomials

a) $z^6 + iz^3 + i - 1 = 0$ b) $(2 - 3i)z^6 + 1 + 5i = 0$

c) $z^{10} + (-2 + i)z^5 - 2i = 0$

Example 12: To show that $z_1 = 1 + i$ is a root of $P(z) = z^{17} + 2z^{15} - 512$ we have to convert z_1 into polar form. Hence, $1 + i = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$. Thence

$$\begin{aligned} z_1^{17} &= \left(\sqrt{2}\right)^{17}(\cos(\pi/4) + i\sin(\pi/4))^{17}, \\ &= \left(\sqrt{2}\right)^{17}(\cos(17\pi/4) + i\sin(17\pi/4)), \\ &= \left(\sqrt{2}\right)^{17}(\cos(\pi/4) + i\sin(\pi/4)). \end{aligned}$$

Similarly we have

$$z_1^1 = \left(\sqrt{2}\right)^{15}(\cos(-\pi/4) + i\sin(-\pi/4)).$$

Therefore

$$\begin{aligned} P(z_1) &= \left(\sqrt{2}\right)^{17}(\cos(\pi/4) + i\sin(\pi/4)) + 2\left[\left(\sqrt{2}\right)^{15}(\cos(-\pi/4) + i\sin(-\pi/4))\right] - 512, \\ &= \left(\sqrt{2}\right)^{15}\left[\left(\sqrt{2}\right)^2\cos(\pi/4) + 2\cos(\pi/4) + i\left(\left(\sqrt{2}\right)^2\sin(\pi/4) - 2\sin(\pi/4)\right)\right] - 512, \\ &= \left(\sqrt{2}\right)^{15}(4\cos(\pi/4)) - 512, \\ &= 4\left(\sqrt{2}\right)^{15}.\frac{\sqrt{2}}{2} - 512 = 0. \end{aligned}$$

Example 13: Suppose a complex number w is located in the first quadrant of the Argand diagram, and is a cube root of a complex number z. Can there exist a second cube root of z located in the first quadrant? Yes. To see this, let $z = x + iy = r(\cos\theta + i\sin\theta)$.

Then

$$\begin{aligned} w = z^{1/3} &= r^{1/3}(\cos\theta + i\sin\theta)^{1/3}, \\ &= r^{1/3}\left(\cos\left(\frac{\theta + 2k\pi}{3}\right) + i\sin\left(\frac{\theta + 2k\pi}{3}\right)\right). \end{aligned}$$

The principal value of this is when $k = 0$. Then $w = r^{1/3}(\cos\theta/3 + i\sin\theta/3)$. When $k = 3$ we obtain another root, say v, where $v = r^{1/3}(\cos(\theta + 6\pi)/3 + i\sin(\theta + 6\pi)/3)$. But this equation simplifies to $v = r^{1/3}(\cos(\theta/3 + 2\pi) + i\sin(\theta/3 + 2\pi))$ which is the same value as w and therefore also lie in the first quadrant of the Argand diagram.

Example 14: Suppose z is a complex number that possesses a fourth root w that is neither real nor purely imaginary. We can show that the remaining fourth roots are also neither real nor purely imaginary. To do this we again let $z = x + iy = r(\cos\theta + i\sin\theta)$, where $\theta \neq k\pi$ and $\theta \neq k\pi/2$, for $k = 0, \pm1, \pm2, \ldots$. Then

$$\begin{aligned} w = z^{1/4} &= r^{1/4}(\cos\theta + i\sin\theta)^{1/4}, \\ &= r^{1/4}\left(\cos\left(\frac{\theta + 2k\pi}{4}\right) + i\sin\left(\frac{\theta + 2k\pi}{4}\right)\right). \\ &= r^{1/4}\left(\cos\left(\frac{\theta}{4} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\theta}{4} + \frac{k\pi}{2}\right)\right). \end{aligned}$$

The principal value of this is when $k = 0$. Then $w = r^{1/4}(\cos\theta/4 + i\sin\theta/4)$ which, by definition, is neither real nor purely imaginary. The three remaining roots are given by

- $r^{1/4}(\cos(\theta/4 + \pi/2) + i\sin(\theta/4 + \pi/2))$ which is perpendicular to w and is therefore neither real nor purely imaginary;
- $r^{1/4}(\cos(\theta/4 + \pi) + i\sin(\theta/4 + \pi))$ which is perpendicular to the previous root and is therefore neither real nor purely imaginary;
- $r^{1/4}(\cos(\theta/4 + 3\pi/2) + i\sin(\theta/4 + 3\pi2))$ which is perpendicular to the previous root and is therefore neither real nor purely imaginary;

1.13.2 Deriving the equation for the roots of a complex number

To derive equation (42**Error! Reference source not found.**) we proceed as follows: Consider a complex number z for which we wish to find the n n^{th} roots. To this end we want to find all roots $z_k = z^{1/n}$, for $k = 0, 1, 2, \ldots, n-1$. We have not yet created the rooting operation, so we do not know how it works. Therefore, let us instead study $z = z_k^n$, since we know how to power complex numbers.

Therefore, let $z = r(\cos\theta + i\sin\theta)$ be the number we wish to take the n^{th} root of. Applying DeMoivre's theorem to $z^{1/n}$ we obtain

$$z^{1/n} = r^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right).$$

This represents only one answer to the root, say z_0. So let $z_0 = R(\cos\phi + i\sin\phi)$. Then $z = z_0^n$, hence

$$r(\cos\theta + i\sin\theta) = R^n(\cos\phi + i\sin\phi)^n = R^n(\cos n\phi + i\sin n\phi)\,. \qquad \{*\}$$

Comparing *Re* and *Im* parts we have $r\cos\theta = R^n\cos n\phi$ and $r\sin\theta = R^n\sin n\phi$. If we square and add these two equations we obtain $r^2(\cos^2\theta + \sin^2\theta) = (R^n)^2(\cos^2 n\phi + \sin^2 n\phi)$ implying $(R^n)^2 = r^2$

$$R = \sqrt[n]{r}\,.$$

where the positive root is taken since R is the modulus of z.

Now, we know that $R^n(\cos n\phi + i\sin n\phi)$ is only one solution to the roots of z. The other solutions are found by accounting for the periodic nature of cos and sin, i.e. $\cos(n\phi + 2k\pi) = \cos\phi$ or $\sin(n\phi + 2k\pi) = \sin\phi$ where $k \in \mathbb{Z}$. Hence {*} becomes

$$r(\cos\theta + i\sin\theta) = R^n(\cos(n\phi + 2k\pi) + i\sin(n\phi + 2k\pi))\,,$$

implying $n\phi + 2k\pi = \theta$, and therefore

$$\phi = \frac{\theta + 2k\pi}{n}.$$

for $k = 0, \pm1, \pm2, \ldots$ (note $2k\pi$ for $k = 0, \pm1, \pm2, \ldots$ is the same as $-2k\pi$ for $k = 0, \mp1, \mp2, \ldots$).

We are now in a position to define the meaning of the rooting operation $z_k = z^{1/n}$, $k = 0, \pm1, \pm2, \ldots$, to be

the n distinct n^{th} roots of a complex number z given by

$$z_k = z^{1/n} = r^{1/n}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right), \qquad k \in \mathbb{Z}. \qquad (43)$$

such that $z_k^n = z$, for $k = 0, 1, 2, \ldots, n-1$, i.e. such that any solutions z_k, when powered by n, give z. In other words, we write our definition for rooting in terms of the powering of complex numbers, and in doing the emphasise is on finding *all* solutions, and not just some solutions.

Since $k \in \mathbb{Z}$ it looks like (43) has an infinite number of solutions. In one sense this is true, but not all these solutions are distinct. All we are interested in at this time are the distinct solutions of (43). To see that there are indeed a limited number of distinct solutions let k_1 and k_2 be integers which differ by a multiple of n, where $n \in \mathbb{Z}$, i.e. $k_2 = k_1 + pn$. Then

$$\frac{\theta + 2k_2\pi}{n} = \frac{\theta + 2\pi(k_1 + pn)}{n} = \frac{\theta + 2k_1\pi}{n} + 2p\pi\,.$$

So $(\theta + 2\pi k_2)/n$ and $(\theta + 2\pi k_1)/n$ differ by multiples of 2π. So in (43) all solutions after $k = 0, 1, 2, \dots, n-1$ are repeats of those based on $k = 0, 1, 2, \dots, n-1$.

1.13.3 Extending exponentiation to rational numbers – The case of irreducible p/q

We have seen how to use DeMoivre's theorem to evaluate roots of the form $(\cos\theta + i\sin\theta)^{1/n}$ for positive integer values of n. Let us now study the use of DeMoivre's theorem to evaluate roots of the form $(\cos\theta + i\sin\theta)^{p/q}$ where p and q are positive integers, and where p/q is irreducible, i.e. in its lowest terms (sometimes described as p and q being co-prime).

Let us start by considering the case of real numbers. For $x = 4$ we know that the positive root is $\sqrt{x} = \sqrt{4} = 2$. We can then cube this result to obtain $2^3 = 8$. On the other hand we could have performed $x^3 = 4^3 = 64$ and then taken the positive root: $\sqrt{64} = 8$. So here we have that $\left(\sqrt{x}\right)^3 = \sqrt{x^3}$, and in general we have $\left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$ when x is real. In other words, the operation of powering and rooting is consistent, whichever way around we perform these operations.

However, having seen in earlier sections that the arithmetic of real numbers does not completely carry over to the arithmetic of complex numbers the question we need to ask is, Is it true that $\left(\sqrt[n]{z}\right)^m = \sqrt[n]{z^m}$ for a complex number z?

To help us answer this question consider $z = 1 + i$. Suppose we want to evaluate $z^{2/5}$. There are three ways in which we can do this using DeMoivre's theorem:

i) The first way is the usual way of firstly accounting for the periodicity of cos and sin, and then applying DeMoivre's theorem:

$$\begin{aligned} z^{2/5} &= \left(\sqrt{2}\right)^{2/5}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{2/5}, \\ \text{So } z^{2/5} &= \left(\sqrt{2}\right)^{2/5}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)\right)^{2/5}, \\ &= \left(\sqrt{2}\right)^{\frac{2}{5}}\left(\cos\frac{2}{5}(\pi/4 + 2k\pi) + i\sin\frac{2}{5}(\pi/4 + 2k\pi)\right). \end{aligned}$$

ii) The second way is to firstly apply DeMoivre's theorem for the squared power, then account for the periodicity of cos and sin, then use DeMoivre's theorem again on the fifth root power:

$$\begin{aligned} z^{2/5} &= \left(\sqrt{2}\right)^{2/5}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{2/5}, \\ &= \left(\sqrt{2}\right)^{2/5}\left(\cos\left(\frac{2\pi}{4}\right) + i\sin\left(\frac{2\pi}{4}\right)\right)^{1/5}, \\ &= \left(\sqrt{2}\right)^{2/5}\left(\cos\left(\frac{2\pi}{4} + 2k\pi\right) + i\sin\left(\frac{2\pi}{4} + 2k\pi\right)\right)^{1/5}, \\ &= \left(\sqrt{2}\right)^{2/5}\left(\cos\left(\frac{2\pi/4 + 2k\pi}{5}\right) + i\sin\left(\frac{2\pi/4 + 2k\pi}{5}\right)\right). \end{aligned}$$

iii) The third way is to apply DeMoivre's theorem on the rational root, and then account for the periodicity of cos and sin:

$$\begin{aligned} z^{2/5} &= \left(\sqrt{2}\right)^{2/5}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{2/5}, \\ &= \left(\sqrt{2}\right)^{2/5}\left(\cos\left(\frac{2\pi}{20}\right) + i\sin\left(\frac{2\pi}{20}\right)\right), \\ &= \left(\sqrt{2}\right)^{2/5}\left(\cos\left(\frac{\pi}{10} + 2k\pi\right) + i\sin\left(\frac{\pi}{10} + 2k\pi\right)\right). \end{aligned}$$

Letting $\cos\theta + i\sin\theta \equiv \operatorname{cis}\theta$ we then obtain the following results for $k = 0, 1, 2, 3, 4$:

	(1)	(2)	(3)
$\boldsymbol{k}$	$\boldsymbol{z_k}$ **(as in i))**	$\boldsymbol{z_k}$ **(as in ii))**	$\boldsymbol{z_k}$ **(as in iii))**
0	$\left(\sqrt{2}\right)^{2/5}\operatorname{cis}\left(\frac{\pi}{10}\right)$	$\left(\sqrt{2}\right)^{2/5}\operatorname{cis}\left(\frac{\pi}{10}\right)$	$(2)^{1/5}\operatorname{cis}\left(\frac{\pi}{10}\right)$
$\boldsymbol{k}$	$\boldsymbol{z_k}$ **(as in i))**	$\boldsymbol{z_k}$ **(as in ii))**	$\boldsymbol{z_k}$ **(as in iii))**
1	$\left(\sqrt{2}\right)^{2/5}\operatorname{cis}\left(\frac{9\pi}{10}\right)$	$\left(\sqrt{2}\right)^{2/5}\operatorname{cis}\left(\frac{5\pi}{10}\right)$	$(2)^{1/5}\operatorname{cis}\left(\frac{21\pi}{10}\right)$
2	$\left(\sqrt{2}\right)^{2/5}\operatorname{cis}\left(\frac{17\pi}{10}\right)$	$\left(\sqrt{2}\right)^{2/5}\operatorname{cis}\left(\frac{9\pi}{10}\right)$	$(2)^{1/5}\operatorname{cis}\left(\frac{41\pi}{10}\right)$
3	$\left(\sqrt{2}\right)^{2/5}\operatorname{cis}\left(\frac{5\pi}{10}\right)$	$\left(\sqrt{2}\right)^{2/5}\operatorname{cis}\left(\frac{13\pi}{10}\right)$	$(2)^{1/5}\operatorname{cis}\left(\frac{61\pi}{10}\right)$
4	$\left(\sqrt{2}\right)^{2/5}\operatorname{cis}\left(\frac{13\pi}{10}\right)$	$\left(\sqrt{2}\right)^{2/5}\operatorname{cis}\left(\frac{17\pi}{10}\right)$	$(2)^{1/5}\operatorname{cis}\left(\frac{81\pi}{10}\right)$

All three columns of z_k in the table above are solutions to $z^{2/5}$. However, only solution in column (2) provide all the roots. The roots in columns (1) and (3) form only a partial set of solutions. The reason for the discrepancy in the set of solutions in (1) and (3) compared to (2) is that the solutions of (1) and (3) makes us jump much further ahead to future roots than do the solutions of (2). Also, note that (3) simply repeats the principal root every 2π, and never specifies any of the other roots.

Therefore, when p/q is irreducible we decide upon the following: for any complex number $z = r(\cos\theta + i\sin\theta)$ we define $z^{p/q}$ as $(z^p)^{1/q}$, i.e.

$$z^{p/q} = (z^p)^{1/q} = r^{p/q}\left(\cos\left(\frac{p\theta + 2k\pi}{q}\right) + i\sin\left(\frac{p\theta + 2k\pi}{q}\right)\right). \tag{44}$$

Equation (44) will be proved in the next section.

<u>*Example 1:*</u> From what we have learnt above we can say that, given $z = \cos 3\pi/4 + i\sin 3\pi/4$, evaluating $z^{7/3} = (\cos 3\pi/4 + i\sin 3\pi/4)^{7/3}$ gives

$$
\begin{aligned}
z^{7/3} &= \left(\cos\left(7\times\frac{3\pi}{4}\right)+i\sin\left(7\times\frac{3\pi}{4}\right)\right)^{1/3}, \\
&= \left(\cos\left(7\times\frac{3\pi}{4}+2k\pi\right)+i\sin\left(7\times\frac{3\pi}{4}+2k\pi\right)\right)^{1/3}, \\
&= \cos\left(\frac{21\pi/4+2k\pi}{3}\right)+i\sin\left(\frac{21\pi/4+2k\pi}{3}\right).
\end{aligned}
$$

Example 2: If $z=i$ we can find $z^{2/3}$ as follows: $r=|z|=1$ and $\theta=\operatorname{Arg}(z)=\pi/2$. Hence

$$z=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\ .$$

Therefore

$$
\begin{aligned}
z^{2/3} &= \left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^{2/3}, \\
&= \left(\cos\left(\frac{2\pi}{2}\right)+i\sin\left(\frac{2\pi}{2}\right)\right)^{1/3}, \\
&= \left(\cos\left(\frac{\pi+2k\pi}{3}\right)+i\sin\left(\frac{\pi+2k\pi}{3}\right)\right).
\end{aligned}
$$

From this we find our roots to be

k	**Root**
0	$z_0=\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)=\frac{1}{2}+i\frac{\sqrt{3}}{2}$
1	$z_1=\cos(\pi)+i\sin(\pi)=-1$
2	$z_2=\cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)=\cos\left(\frac{\pi}{3}\right)-i\sin\left(\frac{\pi}{3}\right)=\frac{1}{2}-i\frac{\sqrt{3}}{2}$

Example 3: To find $z^{3/8}$ when $z=1-i$, we first find the modulus and argument of z to be $r=\sqrt{2}$ and $\theta=-\pi/4$. Hence

$$z^{3/8} = (\sqrt{2})^{3/8}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)^{3/8},$$

$$= \left(\sqrt{2}\right)^{3/8}\left(\cos\left(\frac{3\pi}{4}\right) - i\sin\left(\frac{3\pi}{4}\right)\right)^{1/8},$$

$$= \left(\sqrt{2}\right)^{3/8}\left(\cos\left(\frac{3\pi/4 + 2k\pi}{8}\right) - i\sin\left(\frac{3\pi/4 + 2k\pi}{8}\right)\right).$$

The roots are therefore

k	**Root**
0	$z_0 = \left(\sqrt{2}\right)^{3/8}\left(\cos\left(\frac{3\pi}{32}\right) - i\sin\left(\frac{3\pi}{32}\right)\right)$
1	$z_1 = \left(\sqrt{2}\right)^{3/8}\left(\cos\left(\frac{11\pi}{32}\right) - i\sin\left(\frac{11\pi}{32}\right)\right)$
2	$z_2 = \left(\sqrt{2}\right)^{3/8}\left(\cos\left(\frac{19\pi}{32}\right) - i\sin\left(\frac{19\pi}{32}\right)\right)$
3	$z_3 = \left(\sqrt{2}\right)^{3/8}\left(\cos\left(\frac{27\pi}{32}\right) - i\sin\left(\frac{27\pi}{32}\right)\right)$
4	$z_4 = \left(\sqrt{2}\right)^{3/8}\left(\cos\left(\frac{35\pi}{32}\right) - i\sin\left(\frac{35\pi}{32}\right)\right) = \left(\sqrt{2}\right)^{3/8}\left(\cos\left(\frac{29\pi}{32}\right) + i\sin\left(\frac{29\pi}{32}\right)\right)$
5	$z_5 = \left(\sqrt{2}\right)^{3/8}\left(\cos\left(\frac{43\pi}{32}\right) - i\sin\left(\frac{43\pi}{32}\right)\right) = \left(\sqrt{2}\right)^{3/8}\left(\cos\left(\frac{21\pi}{32}\right) + i\sin\left(\frac{21\pi}{32}\right)\right)$
6	$z_6 = \left(\sqrt{2}\right)^{3/8}\left(\cos\left(\frac{51\pi}{32}\right) - i\sin\left(\frac{51\pi}{32}\right)\right) = \left(\sqrt{2}\right)^{3/8}\left(\cos\left(\frac{13\pi}{32}\right) + i\sin\left(\frac{13\pi}{32}\right)\right)$
7	$z_7 = \left(\sqrt{2}\right)^{3/8}\left(\cos\left(\frac{59\pi}{32}\right) - i\sin\left(\frac{59\pi}{32}\right)\right) = \left(\sqrt{2}\right)^{3/8}\left(\cos\left(\frac{5\pi}{32}\right) + i\sin\left(\frac{32\pi}{32}\right)\right)$

<u>*Example 4:*</u> To find the sum and product of the first five values of $(\cos\pi + i\sin\pi)^{2/5}$ we proceed as follows: Let $z = \cos\pi + i\sin\pi$. Then

$$z^{2/5} = (\cos\pi + i\sin\pi)^{2/5} = \cos\left(\frac{2\pi + 2k\pi}{5}\right) + i\sin\left(\frac{2\pi + 2k\pi}{5}\right).$$

Hence the first five values are

- for $k = 0$, $z_0 = \cos(2\pi/5) + i\sin(2\pi/5)$;
- for $k = 1$, $z_1 = \cos(4\pi/5) + i\sin(4\pi/5)$;

- for $k = 2, z_2 = \cos(6\pi/5) + i\sin(6\pi/5) = \cos(4\pi/5) - i\sin(4\pi/5)$;
- for $k = 3, z_3 = \cos(8\pi/5) + i\sin(8\pi/5) = \cos(2\pi/5) - i\sin(2\pi/5)$;
- for $k = 4, z_4 = \cos(10\pi/5) + i\sin(10\pi/5) = 1$;

Given our previous theory, note that we are not taking the two-fifths roots of $z = \cos\pi + i\sin\pi$, but the fifth roots of the new complex number $(w =)\ z^2 = \cos 2\pi + i\sin 2\pi$.

Since $z^2 = \cos 2\pi + i\sin 2\pi = 1$, and since we have found the fifth roots of w, we are in fact solving the polynomial $w^5 - 1 = 0$. By standard algebra we know that the sum of roots of a real polynomial $ax^n + bx^{n-1} \ldots + a_0 = 0$ is given by $-b/a$. Hence for $w^5 - 1 = 0$ we have the sum of the roots to be zero , therefore

$$z_0 + z_1 + z_2 + z_3 + z_4 = 1 + 2\cos(2\pi/5) + 2\cos(4\pi/5) = 0\,.$$

For the product of roots we have

$$[\cos(2\pi/5) + i\sin(2\pi/5)] \times [\cos(4\pi/5) + i\sin(4\pi/5)] \times [\cos(-4\pi/5) + i\sin(-4\pi/5)]$$

$$\times [\cos(-2\pi/5) + i\sin(-2\pi/5)] \times 1\,.$$

By the property that multiplication of complex numbers equals addition of their arguments, we have

$$\cos\left(\frac{2\pi}{5} + \frac{4\pi}{5} + \frac{6\pi}{5} + \frac{8\pi}{5} + \frac{10\pi}{5}\right) + i\sin\left(\frac{2\pi}{5} + \frac{4\pi}{5} + \frac{6\pi}{5} + \frac{8\pi}{5} + \frac{10\pi}{5}\right) = \cos(6\pi) = 1\,.$$

<u>*Example 5:*</u>

To find the principal value of $(1 + \cos\theta + i\sin\theta)^{3/4}$ in $-\pi < \theta \leq \pi$, and in $\pi < \theta \leq 3\pi$, we first have to convert $1 + \cos\theta + i\sin\theta$ into the correct form so that we can use DeMoivre's theorem. There are (at least) two ways in which we can do this, both of which illustrate certain points worth knowing about. So we will go through two different solutions to this problems in order to highlight these points.

<u>*Solution 1*</u>

We can transform $1 + \cos\theta$ and $\sin\theta$ using standard trig identities. From $\cos 2\theta = 2\cos^2\theta - 1$ we have $1 + \cos\theta = 2\cos^2\theta/2$, and from $\sin 2\theta = 2\sin\theta\cos\theta$ we have $\sin\theta = 2\sin\theta/2\cos\theta/2$. Hence

$$
\begin{aligned}
1+\cos\theta+i\sin\theta &= 2\cos^2\theta/2+2i\sin\theta/2\cos\theta/2\,, \\
&= 2\cos\theta/2\,(\cos\theta/2+i.\sin\theta/2)\,. \qquad (*)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(1+\cos\theta+i\sin\theta)^{3/4} &= \big(2\cos(\theta/2)\,(\cos\theta/2+i.\sin\theta/2)\big)^{3/4}\,, \\
&= \left(2\cos\frac{\theta}{2}\right)^{\frac{3}{4}}\left(\cos\frac{3\theta}{8}+i.\sin\frac{3\theta}{8}\right).
\end{aligned}
$$

This solution is the principal value of $(1+\cos\theta+i\sin\theta)^{3/4}$ in $-\pi<\theta\leq\pi$.

For the case of $\pi<\theta\leq 3\pi$ we must first shift the period of the trig functions from $-\pi<\theta\leq\pi$ to $\pi<\theta\leq 3\pi$ so that the principal argument now lies within $\pi<\theta\leq 3\pi$. This means that the argument of our previous solution needs to be shifted along by 2π. In other words, we must first be in the correct “principal value” interval before taking roots.

Now, since $\cos(\theta+2k\pi)=\cos\theta$, we therefore have $\cos(\theta/2+k\pi)=\cos\theta/2$, and equation (*) becomes

$$
1+\cos\theta+i\sin\theta = 2\cos(\theta/2+k\pi)\,(\cos(\theta/2+k\pi)+i.\sin(\theta/2+k\pi))\,.
$$

The RHS of the above equation is the general form of $1+\cos\theta+i\sin\theta$, and can be used to find the principal value in any appropriate interval. For the case of $-\pi<\theta\leq\pi$ we set $k=0$. In our current case we set $k=1$ to find the principal value of $1+\cos\theta+i\sin\theta$ in $\pi<\theta\leq 3\pi$. Hence

$$
\begin{aligned}
(1+\cos\theta+i\sin\theta)^{3/4} &= \big(2\cos(\theta/2+\pi)\,(\cos(\theta/2+\pi)+i.\sin(\theta/2+\pi))\big)^{3/4}\,, \\
&= \left(2\cos\left(\frac{\theta}{2}+\pi\right)\right)^{\frac{3}{4}}\left(\cos\frac{3}{4}\left(\frac{\theta}{2}+\pi\right)+i.\sin\frac{3}{4}\left(\frac{\theta}{2}+\pi\right)\right), \\
&= \left(-2\cos\frac{\theta}{2}\right)^{\frac{3}{4}}\left(\cos\frac{3}{8}(\theta+2\pi)+i.\sin\frac{3}{8}(\theta+2\pi)\right).
\end{aligned}
$$

<u>*Solution 2*</u>

In solution 1 we had to be aware of shifting the argument appropriately in order for it to lie in the interval we wanted. In this solution we will not need to manually shift the argument since this will automatically be taken into account by the alternative way in which we solve the problem.

To this end, notice that $1 = \cos 0 + i \sin 0$ for the interval $-\pi < \theta \leq \pi$, and we could repeat solution 1 with this transformation, resulting in the same answer (left as exercise). In the case of $\pi < \theta \leq 3\pi$ we set $1 = \cos 2\pi + i \sin 2\pi$. Hence, we can alternatively transform $1 + \cos\theta + i\sin\theta$ as follows:

$$\begin{aligned} 1 + \cos\theta + i\sin\theta &= \cos 2\pi + i\sin 2\pi + \cos\theta + i\sin\theta\,, \\ &= \cos 2\pi + \cos\theta + i(\sin 2\pi + \sin\theta)\,. \end{aligned}$$

What we want to do is convert this last expression using the factor formula from the trig family of indentities, to obtain

$$1 + \cos\theta + i\sin\theta = 2\cos\left(\pi + \frac{\theta}{2}\right).\cos\left(\pi - \frac{\theta}{2}\right) + 2i\sin\left(\pi + \frac{\theta}{2}\right).\cos\left(\pi - \frac{\theta}{2}\right)\,,$$

from which we can factor out the term $2\cos(\pi - \theta/2)$ to get

$$1 + \cos\theta + i\sin\theta = \left(2\cos\left(\pi - \frac{\theta}{2}\right)\right)\left(\cos\left(\pi + \frac{\theta}{2}\right) + i\sin\left(\pi + \frac{\theta}{2}\right)\right).$$

Hence

$$\begin{aligned} (1 + \cos\theta + i\sin\theta)^{3/4} &= \left(2\cos\left(\pi - \frac{\theta}{2}\right)\right)^{3/4}\left(\cos\left(\pi + \frac{\theta}{2}\right) + i\sin\left(\pi + \frac{\theta}{2}\right)\right)^{3/4}, \\ &= \left(-2\cos\frac{\theta}{2}\right)^{\frac{3}{4}}\left(\cos\frac{3}{4}\left(\pi + \frac{\theta}{2}\right) + i\sin\frac{3}{4}\left(\pi + \frac{\theta}{2}\right)\right), \\ &= \left(-2\cos\frac{\theta}{2}\right)^{\frac{3}{4}}\left(\cos\frac{3}{8}(\theta + 2\pi) + i.\sin\frac{3}{8}(\theta + 2\pi)\right). \end{aligned}$$

1.13.4 Deriving DeMoivre's theorem for irreducible an exponent p/q

Let us return to the idea of the rooting of positive real numbers. As such, consider $x = 8$. We know that $x^{2/3} = 8^{2/3} = 4$. We also know ths this can be performed in two different ways, both of which give the same answer: $\left(8^{1/3}\right)^2 = 2^2 = 4$ or $(8^2)^{1/3} = 64^{1/3} = 4$. In other words it doesn't matter whether we perform the squaring first, and then the rooting, or vice-versa.

However, having seen in earlier sections that the arithmetic of real numbers does not completely carry over to the arithmetic of complex numbers the question we need to ask is, Is it true that $z^{p/q} = \left(z^{1/q}\right)^p = (z^p)^{1/q}$ for a complex number z? This is what we will now address.

Here we will prove equation (44) for the case of irreducible p/q, where $p, q \in \mathbb{N}$. The logic behind this proof is identical to that we have previously used in section 1.13.2 when proving $z^{1/n}$ via Demoivre's theorem.

So we proceed as follows: Consider a complex number z for which we wish to find the p/q^{th} roots. To this end we want to find all roots $z_k = z^{p/q}$, for $k = 0, 1, 2, \dots, n-1$. Our aim is to find all distinct roots, before repetition due to periodicity $2k\pi$, and we do this by analysing $z^p = z_k^q$. We therefore say that we want all solutions z_k such that $z^p = z_k^q$. This then guarantees us finding *all* roots z_k.

Therefore, let $z = r(\cos\theta + i\sin\theta)$ be the number we wish to take the p/q^{th} root of. By DeMoivre's theorem we have

$$z^{p/q} = r^{p/q}(\cos\theta + i\sin\theta)^{p/q} = r^{p/q}\left(\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}\right).$$

The RHS of this equation represents only one value, say z_0. So let $z_0 = R(\cos\phi + i\sin\phi)$. Then $z^{p/q} = z_0$ implies $z^p = z_0^q$, hence

$$r^p(\cos p\theta + i\sin p\theta) = R^q(\cos\phi + i\sin\phi)^q = R^q(\cos q\phi + i\sin q\phi)\,. \qquad [*]$$

Comparing *Re* and *Im* parts we have $r^p\cos p\theta = R^q\cos q\phi$ and $r^p\sin p\theta = R^q\sin q\phi$. If we square and add these two equations we obtain $(r^p)^2(\cos^2 p\theta + \sin^2 p\theta) = (R^q)^2(\cos^2 q\phi + \sin^2 q\phi)$ implying $(R^q)^2 = (r^p)^2$. Hence

$$R = \sqrt[q]{r^p}\,.$$

where the positive root is taken since R is the modulus of z.

Now, we know that $R^q(\cos q\phi + i\sin q\phi)$ is only one solution to the roots of z. The other solutions are found by accounting for the periodic nature of cos and sin, i.e. $\cos(q\phi + 2k\pi) = \cos q\phi$ or $\sin(q\phi + 2k\pi) = \sin q\phi$ where $k \in \mathbb{Z}$. Hence [*] becomes

$$r^p(\cos p\theta + i\sin p\theta) = R^q(\cos(q\phi + 2k\pi) + i\sin(q\phi + 2k\pi))\,,$$

implying $q\phi + 2k\pi = p\theta$, and therefore

$$\phi = \frac{p\theta + 2k\pi}{q}\,.$$

for $k = 0, \pm1, \pm2, \dots$ (note $2k\pi$ for $k = 0, \pm1, \pm2, \dots$ is the same as $-2k\pi$ for $k = 0, \mp1, \mp2, \dots$).

We are now in a position to define the meaning of the rooting operation $z_k = z^{p/q}$, $k = 0, \pm 1, \pm 2, \ldots$, to be

the q distinct q^{th} roots of a complex number z^p given by

$$z_k = z^{p/q} = (z^p)^{1/q} = r^{p/q}\left(\cos\left(\frac{p\theta + 2k\pi}{q}\right) + i\sin\left(\frac{p\theta + 2k\pi}{q}\right)\right), \qquad k \in \mathbb{Z}. \tag{45}$$

such that $z^p = z_k^q$, for $k = 0, 1, 2, \ldots, q-1$,

i.e. such that any solutions z_k, when powered by q, give z^p.

1.14 Issues when finding roots of a complex number

1.14.1 Issue 1: An inconsistency in DeMoivre's theorem – The rooting operations gives different results

We know that, for real numbers, adding 2 to 3 gives one unique result: $3 + 2 = 5$. There is no other answer but 5 when we do this operation. The same is true for subtraction, multiplication, division, and exponentiation:

- $3 - 2 = 1$, and there is no other answer but 1 when we do this operation;
- $2 \times 3 = 6$, and there is no other answer but 6 when we do this operation;
- $6 \div 3 = 2$, and there is no other answer but 2 when we do this operation;
- $3^2 = 9$, and there is no other answer but 9 when we do this operation;

Similarly for complex numbers we have

- $(1 + i) + (3 - 2i) = 4 - i$, and there is no other answer but $4 - i$ when we do this operation;
- $(1 + i) - (3 - 2i) = -2 + 3i$, and there is no other answer but $-2 + 3i$ when we do this operation;
- $(1 + i)(3 - 2i) = 3 + i$, and there is no other answer but $3 + i$ when we do this operation;
- $(1 + i) \div (3 - 2i) = (1 + 5i)/13$, and there is no other answer but $(1 + 5i)/13$ when we do this operation;
- $(1 + i)^2 = 2i$, and there is no other answer but $2i$ when we do this operation.

Now come things like the square root of positive numbers. For example, $\sqrt{4} = 2$. However, this is not the only answer to $\sqrt{4}$ since we also have $\sqrt{4} = -2$. The reason for this is that both 2 and -2, when squared, give the answer 4. So, the operation of square rooting does not give a unique answer.

As we have seen above, this also happens to be the case when taking roots of complex numbers. For example, the value of $z^{1/2}$ when $z = 2i$ is $z_0 = \sqrt{2}(\cos \pi/4 + i \sin \pi/4) = 1 + i$. But we also know that $z_1 = \sqrt{2}(\cos 5\pi/4 + i \sin 5\pi/4) = -1 - i$ is another root of z. So the same operation of rooting gives us two different answers implying that DeMoivre's theorem is not consistent in the case of rational powers. This is an important point to note.

Another form of inconsistency in the use of DeMoivre's theorem has been illustrated previously with respect to taking account of the periodicty of solutions during the process of taking roots of a complex number. Letting $\cos\theta + i \sin\theta$ be represented by $\operatorname{cis}\theta$, we have for $z^{1/2} = i^{1/2}$ either $z_k = \operatorname{cis}\big((\pi/2 + 2k\pi)/2\big)$ or $w_k = \operatorname{cis}\big((\pi/2)/2 + 2k\pi\big)$. Illustrating this analysis for $i^{1/2}$, $i^{1/3}$ and $i^{1/4}$, we obtain the results in the table below

$\boldsymbol{i^{1/2}}$ **:**	$z_0 = \operatorname{cis}\left(\frac{\pi}{4}\right)$,	$z_1 = \operatorname{cis}\left(\frac{5\pi}{4}\right)$,	$z_2 = \operatorname{cis}\left(\frac{9\pi}{4}\right)$.		
	$w_0 = \operatorname{cis}\left(\frac{\pi}{4}\right)$,	$w_1 = \operatorname{cis}\left(\frac{9\pi}{4}\right)$.			
$\boldsymbol{i^{1/3}}$ **:**	$z_0 = \operatorname{cis}\left(\frac{\pi}{6}\right)$,	$z_1 = \operatorname{cis}\left(\frac{5\pi}{6}\right)$,	$z_2 = \operatorname{cis}\left(\frac{9\pi}{4}\right)$,	$z_3 = \operatorname{cis}\left(\frac{13\pi}{6}\right)$.	
	$w_0 = \operatorname{cis}\left(\frac{\pi}{6}\right)$,	$w_1 = \operatorname{cis}\left(\frac{13\pi}{6}\right)$,	$w_2 = \operatorname{cis}\left(\frac{25\pi}{6}\right)$.		
$\boldsymbol{i^{1/4}}$ **:**	$z_0 = \operatorname{cis}\left(\frac{\pi}{8}\right)$,	$z_0 = \operatorname{cis}\left(\frac{5\pi}{8}\right)$,	$z_2 = \operatorname{cis}\left(\frac{9\pi}{8}\right)$,	$z_3 = \operatorname{cis}\left(\frac{13\pi}{8}\right)$.	$z_4 = \operatorname{cis}\left(\frac{17\pi}{8}\right)$.
	$w_0 = \operatorname{cis}\left(\frac{\pi}{8}\right)$,	$w_1 = \operatorname{cis}\left(\frac{17\pi}{8}\right)$,	$w_2 = \operatorname{cis}\left(\frac{33\pi}{8}\right)$,	$w_4 = \operatorname{cis}\left(\frac{49\pi}{8}\right)$.	

By looking at the relevant colour coordinated results we can see that the principal values are the same, but that, as we take higher and higher roots, results derived from $\mathrm{cis}(\theta/n + 2k\pi)$ miss out more and more roots compared to those derived from $\mathrm{cis}[(\theta + 2k\pi)/n]$.

All of this leads us to defining the n^{th} roots of a complex number in the very particular way of

those complex numbers z_k, for $k = 0, 1, 2, \ldots, n-1$,
which satisfy $(z_k)^n = z$.

Stating the definition in this form says that we are looking for *any and all roots* of the original complex number. We still calculate z_k as $z^{1/n}$ or as $(z^p)^{1/q}$, but we do so in a way that gives us all posible roots (i.e. by the proper accounting of periodicity).

So, one thing to notice in general is that when extending the domain of numbers from $\mathbb{R}$ to $\mathbb{C}$ we gain the ability to find roots that we can't find in $\mathbb{R}$, but we lose the consistency in the use of exponents.

1.14.2 Issue 2: An inconsistency in DeMoivre's theorem – The case of reducible p/q

Let us return to the idea of the rooting of positive real numbers. As such, consider $x = 4$. We know that the positive root is $x^{1/2} = 4^{1/2} = 2$. We also know that $x^{2/4} = 4^{2/4} = 4^{3/6} = 2$. In other words it doesn't matter whether our fractional exponent is in its lowest terms or not.

However, having seen in earlier sections that the arithmetic of real numbers does not completely carry over to the arithmetic of complex numbers the question we need to ask is, Is it true that $z^{p/q} = z^{2p/2q} = \cdots = z^{kp/kq}$ for a complex number z? This is what we will now address.

Therefore, consider evaluating $1^{2/2}$. This can be

As another example, consider finding the square root of $z = 1 + i$. Doing so gives

$$\begin{aligned} z^{1/2} &= \left(\sqrt{2}\right)^{1/2}(\cos(\pi/4 + 2k\pi) + i\sin(\pi/4 + 2k\pi))^{1/2}, \\ &= \left(\sqrt{2}\right)^{1/2}\left(\cos\left(\frac{\pi/4 + 2k\pi}{2}\right) + i\sin\left(\frac{\pi/4 + 2k\pi}{2}\right)\right). \end{aligned}$$

Now consider finding the fourth roots of $w = 2i$. Doing so gives

$$
\begin{aligned}
w^{1/4} &= (2)^{1/4}(\cos(\pi/2 + 2k\pi) + i\sin(\pi/2 + 2k\pi))^{1/4}\,, \\
&= (2)^{1/4}\left(\cos\left(\frac{\pi/2 + 2k\pi}{4}\right) + i\sin\left(\frac{\pi/2 + 2k\pi}{4}\right)\right).
\end{aligned}
$$

The individual roots of w and z are then as follows:

$\boldsymbol{k}$	$\boldsymbol{w_k}$	$\boldsymbol{z_k}$
0	$(2)^{1/4}(\cos\pi/8 + i\sin\pi/8)$	$\left(\sqrt{2}\right)^{1/2}(\cos\pi/8 + i\sin\pi/8)$
1	$(2)^{1/4}(\cos 5\pi/8 + i\sin 5\pi/8)$	$\left(\sqrt{2}\right)^{1/2}(\cos 9\pi/8 + i\sin 9\pi/8)$
2	$(2)^{1/4}(\cos 9\pi/8 + i\sin 9\pi/8)$	
3	$(2)^{1/4}(\cos 13\pi/8 + i\sin 13\pi/8)$	

Now note that $w = z^2$, so what we have done in finding the fourth roots of w is to find the fourth roots of z^2, i.e. $(z^2)^{1/4}$. We know that, in terms of real number arithmetic, $2/4 = ½$. But we can now see from the table above that, as a complete solution, $w^{1/4} = (z^2)^{1/4} \neq z^{1/2}$. Hence $z^{2/4} \neq z^{1/2}$, and in general we have that if p and q are not coprime (where $p, q \in \mathbb{N}$) then $(z^p)^{1/q} \neq z^{p/q}$.

Therefore, for any complex number $z = r(\cos\theta + i\sin\theta)$ we decide upon the following: when p/q is in reducible form we firstly apply Demoivre's theorem on the power p, then we take account of the periodicity of the trig functions, then we take the q^{th} roots. In other words,

$$
\begin{aligned}
z^{p/q} &= r^{p/q}\{\cos\theta + i\sin\theta\}^{p/q}\,, \\
&= r^{p/q}\{\cos p\theta + i\sin p\theta\}^{1/q}\,,
\end{aligned}
$$

this latter equation being a new complex number w whose argument is $\phi = p\theta$, after which we apply DeMoivre's theorem again to find the q^{th} roots of w:

$$
\begin{aligned}
w^{1/q} = (z^p)^{1/q} &= r^{p/q}\{\cos(\phi + 2k\pi) + i\sin(\phi + 2k\pi)\}^{1/q}\,, \\
&= r^{p/q}\left\{\cos\left(\frac{\phi}{q} + \frac{2k\pi}{q}\right) + i\sin\left(\frac{\phi}{q} + \frac{2k\pi}{q}\right)\right\}.
\end{aligned}
$$

which is equivalent to

$$z^{p/q} = r^{p/q}\left\{\cos\left(\frac{p\theta}{q} + \frac{2k\pi}{q}\right) + i\sin\left(\frac{p\theta}{q} + \frac{2k\pi}{q}\right)\right\}. \tag{46}$$

(remember that, since r is a real number, $(r^p)^{1/q} = r^{p/q}$). Note that the interval for ϕ is $-\pi < \phi \leq \pi$, implyings $-\pi < p\theta \leq \pi$. So the correct interval for principal arguments with respect to θ is $-\pi/p < \theta \leq \pi/p$.

After all this work we can now define the p/q^{th} roots of a complex number of $z^{p/q}$ to be the q^{th} roots of z^p, this being

> those complex numbers z_k, for $k = 0, 1, 2, \dots, n-1$,
> which satisfy $(z_k)^q = z^p$.

<u>*Exercise:*</u> If $z = 2 + 2i$ and $w = z^3$ find $w^{1/2}$ and $z^{3/6}$.

Normally when we find roots of a complex number z we are find these roots with respect to z itself. However, in the example above we end up finding the fourth roots of z with respect to z^2. So in (46) we are finding the q^{th} roots of z with respect to z^p, not with respect to the original complex number z.

1.14.3 Issue 3: The non-distributive nature of taking roots of a complex number

Here we will adress in more detail what is going on when we “prove” of $1 = -1$, as was illustrated in section 1.2. Before getting to this we will look at the operation opposite to rooting, namely powering.

Therefore, consider $z = 2$. This number lies on the positive *Re* axis. In polar form this number is

$$z = 2(\cos 0 + i\sin 0)\,.$$

Squaring this value gives

$$z^2 = 4(\cos 0 + i\sin 0)^2\,.$$

Since sin and cos are periodic function we can write this number as

$$z^2 = 4(\cos 2k\pi + i\sin 2k\pi)^2\,,$$

for $k = 0, \pm1, \pm2, \dots$

Applying DeMoivre's theorem gives

$$z^2 = 4(\cos 4k\pi + i\sin 4k\pi)\ .$$

What this shows is that, even though we have multiple answers to z^2 (actually, an infinite number of answers) *they are all the same value*:

$$\begin{array}{rcl} \ldots & = & 4 \\ z = 4(\cos(-8\pi) + i\sin(-8\pi)) & = & 4 \\ z = 4(\cos(-4\pi) + i\sin(-4\pi)) & = & 4 \\ z = 4(\cos 0 + i\sin 0) & = & 4 \\ z = 4(\cos(4\pi) + i\sin(4\pi)) & = & 4 \\ z = 4(\cos(8\pi) + i\sin(8\pi)) & = & 4 \\ \ldots & = & 4 \end{array}$$

So it doesn't matter which k value we choose in $4(\cos 4k\pi + i\sin 4k\pi)$ since we will always obtain the same result.

But if we now consider the number $z = 4$, whose polar form is $z = 4(\cos 0 + i\sin 0)$, and we wish to take the square root of z we have, as usual,

$$z^{1/2} = 2(\cos 2k\pi + i\sin 2k\pi)^{1/2} = 2(\cos k\pi + i\sin k\pi)$$

for $k = 0, \pm 1, \pm 2, \ldots$ in this case we have the following values

$$\begin{array}{rcl} \ldots & = & 2 \\ z = 2(\cos(-\pi) + i\sin(-\pi)) & = & -2 \\ z = 2(\cos(0) + i\sin(0)) & = & 2 \\ \boldsymbol{z = 2(\cos \pi + i\sin \pi)} & = & \boldsymbol{-2} \\ z = 2(\cos(2\pi) + i\sin(2\pi)) & = & 2 \\ z = 2(\cos(3\pi) + i\sin(3\pi)) & = & -2 \\ \ldots & = & 2 \end{array}$$

where the value in bold is the principal value. So in the case of rooting it does indeed matter which k value we choose in $2(\cos 4k\pi + i\sin 4k\pi)$ since we get different answers for different values of k.

In summary we therefore have

- a real number, when raised to a positive integer power, gives *multiple answers which are all the same*, and therefore always lie on the positive *Re* axis, and at the same location;
- a real number, when rooted, gives *multiple answers which are different*, and which are therefore distributed (by varying angles) to different locations across the complex plane.

This leads us to having to ask the following: when rooting, which root are we actually finding, and which root do we want to refer to?

Let us now turn to the "proof" of $1 = -1$ of section 1.2.2. One step in this "proof" was $\sqrt{(-1)(-1)} = \sqrt{-1} \times \sqrt{-1}$. Calculating $\sqrt{(-1)(-1)}$ we have

$$
\begin{aligned}
1 = \sqrt{(-1)(-1)} &= [(\cos\pi + i\sin\pi)(\cos\pi + i\sin\pi)]^{1/2}, \\
&= (\cos 2\pi + i\sin 2\pi)^{1/2}\ , \\
&= (\cos(2\pi + 2k\pi) + i\sin(2\pi + 2k\pi))^{1/2}\ , \\
&= (\cos(\pi + k\pi) + i\sin(\pi + k\pi))\ , \\
&= -1 \text{ for } k = 0. \\
&= +1 \text{ for } k = 1.
\end{aligned}
$$

Calculating $\sqrt{-1} \times \sqrt{-1}$ we have

$$
\begin{aligned}
\sqrt{(-1)} \times \sqrt{(-1)} &= (\cos\pi + i\sin\pi)^{1/2}(\cos\pi + i\sin\pi)^{1/2}\ , \\
&= (\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi))^{1/2} \\
&\qquad\qquad \times (\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi))^{1/2} \\
&= \left(\cos\left(\frac{\pi}{2} + k\pi\right) + i\sin\left(\frac{\pi}{2} + k\pi\right)\right)\left(\cos\left(\frac{\pi}{2} + k\pi\right) + i\sin\left(\frac{\pi}{2} + k\pi\right)\right), \\
&= \cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)\ , \\
&= -1 \text{ for } k = 0, \\
&= -1 \text{ for } k = 1.
\end{aligned}
$$

So, in the first solution, where we did not distribute the root, we get two answers and in the second solution, where we did distribute the root, we only get one answer. So there is an inconsistency between the the number of solutions produced, and it is for this reason that we cannot distribute the roots across negative numbers (notice that the inconsistency arises because, in solution 1, we have accounted for the periodicity $2k\pi$ only once, whereas in solution 2 we have accounted for the periodicity $2k\pi$ twice, once for each complex number).

Now, although the answer to solution 2 is correct, solution 2 produces an incomplete set of answers. And it is the aim of maths to solve problems in such as way as to produce all possible answers to a problem. Many hundreds of years of the evolution of maths lead to a period from the mid 1800s to the beginning of the 1900s (and giving rise to the discipline of real analysis) which culminated in a way of defining answers to problems as the set of all possible solutions, not simply the set of some possible answers. Solution 1 above produces the set of all possible solutions.

Therefore, the question is, How do we define the rooting problem so that we do not end up with the issue above? Well, we do this by defining it as a squaring problem. To see why, notice that the solutions ± 1 can be seen to be the result of solving the square root problem $z = \sqrt{(-1)(-1)}$. However, we can restate this problem as that of wanting to solve the squaring problem $z^2 = 1$. What this means that we want square roots not as answers to solving the square root problem (which may or may not produce a complete set of answers depending on how we take the square root), but as answers which satisfy the squaring problem $x^2 = 1$. And we do this by saying

$$\sqrt{(-1)(-1)} = \pm 1 \textit{ because } \left(\sqrt{(-1)(-1)}\right)^2 = (\pm 1)^2, \text{ i.e. } 1 = 1.$$

I.e. we have to define the taking of roots as that process which gives us all solution to the power problem. Therefore, in general we don't just define rooting as $z_k = (z)^{1/2}$ but as $z_k = (z)^{1/2}$ *such that* $z_k^2 = z$, i.e. as all solution z_k which can be squared to give z.

Such an approach to solving the rooting problem then forces us to look for a method of rooting which produces the maximal numbers of roots which satisfy $x^2 = 1$. It also means we no longer need to worry about which of the two answers to choose. We no longer consider $\sqrt{-1} = i$ and/or $\sqrt{-1} = -i$. Instead we consider $\left(\sqrt{-1}\right)^2 = i^2 = -1$, i.e. an equation for which it is clear that there is only one answer, namely the value -1.

Having then found all the necessary roots to $\sqrt{(-1)(-1)}$ how are we to sort out the problem of $1 = -1$? Well, the short answer is that we need to keep mathematics consistent, so given two answers to our roots we choose that answer which keeps mathematics consistent. In other words, between a choice of $+1$ and -1 as our answer to $\sqrt{(-1)(-1)}$ we clearly have to choose $+1$ in order to have $1 = \sqrt{(-1)(-1)} = +1$. Similarly, the only possible way $-1 = \sqrt{(-1)(-1)}$ is if we choose the answer to $\sqrt{(-1)(-1)}$ as being -1. Then we can say $-1 = \sqrt{(-1)(-1)} = -1$

Hence, in general, given two complex numbers z_1 and z_2, and a positive integer n, we have

$$(z_1 z_2)^n = z_1^n . z_2^n \qquad \text{but} \qquad (z_1 z_2)^{1/n} \neq z_1^{1/n} . z_2^{1/n}$$

One thing to notice in general is that when extending the domain of numbers from $\mathbb{R}$ to $\mathbb{C}$ we gain the ability to operate on the roots of negative numbers, but we lose the flexibility of distributing roots across multiplication.

As an exercise try creating a proof that $1 = 2$ by incorrectly distributing the square root (hint: start with $(-1)/1 = 1/(-1)$ and later on add $3/(2i)$ to both sides of the equation).

1.15 On the roots of unity of a complex number

We are all familiar with the fact that if $x^2 = 1$ then $x = \pm 1$. But what if $x^3 = 1$? Since this is a cubic we expect three roots, but it seems that only $x = 1$ satisfies this equation. What about the roots of $x^4 = 1$? We know that $x = \pm 1$ satisfy this equation, but since $x^4 = 1$ is a quartic equation we should have another two roots. Where are they?

More generally, if $x^n = 1$ we expect n roots as solutions, but it seems we can find only a maximum of two roots ($x = 1$ or $x = \pm 1$), depending on whether n is even or odd. The resolution to this apparent problem is found in complex analysis. The idea of complex numbers can now be used to find the roots of unity, i.e. the roots of the number 1.

1.15.1 The structure of the roots of unity

Let us first start our study by analysing the cube roots of $x^3 = 1$. Since this implies $x^3 - 1 = 0$ we can factorise this as

$$x^3 - 1 = (x - 1)(x^2 + x + 1) = 0 .$$

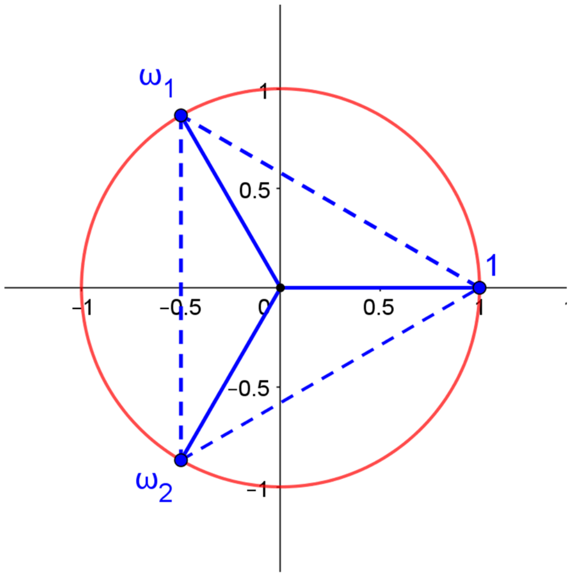
So one root is $\omega_0 = 1$. By the quadratic fomula we get the following roots for $x^2 + x + 1 = 0$:

$$\omega_1 = \frac{-1 + i\sqrt{3}}{2} \text{ and } \omega_2 = \frac{-1 - i\sqrt{3}}{2}.$$

Therefore $x^3 - 1 = 0$ can be factorised as

$$x^3 - 1 = (x - 1)\left(x - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right)\left(x - \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right).$$

We can plot these on an Argand diagram as illustrated below. Notice that the roots lie on the circumference of a unit circle, and also form the vertices of triangle:



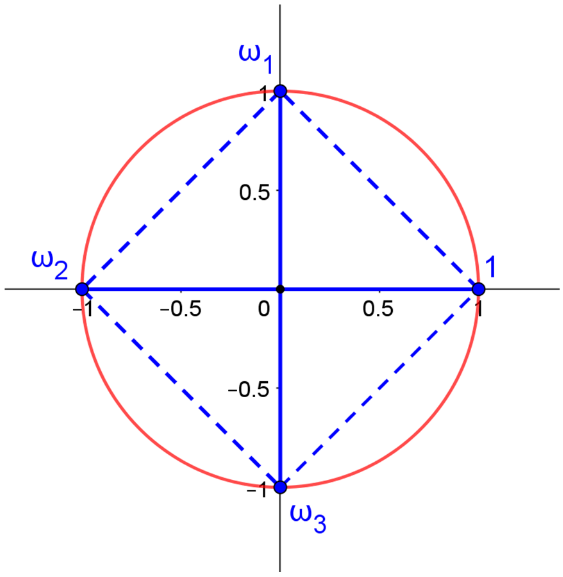
Repeating the analysis for the fourth roots of $x^4 = 1$ we obtain

$$x^4 - 1 = (x - 1)(x + 1)(x^2 + 1) = 0\,,$$

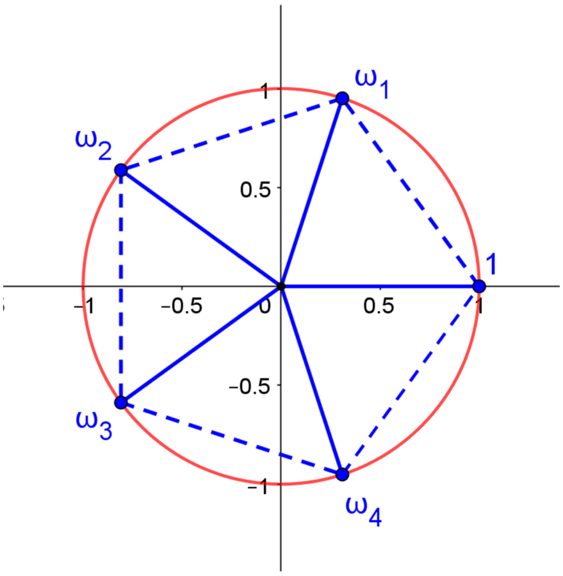
giving roots roots (taken in order) of $\omega_0 = 1$, $\omega_1 = i$, $\omega_2 = -1$ and $\omega_3 = -i$. Hence $x^4 - 1 = 0$ can be factorised as

$$x^4 - 1 = (x - 1)(x + 1)(x - i)\big(x - (-i)\big)\,.$$

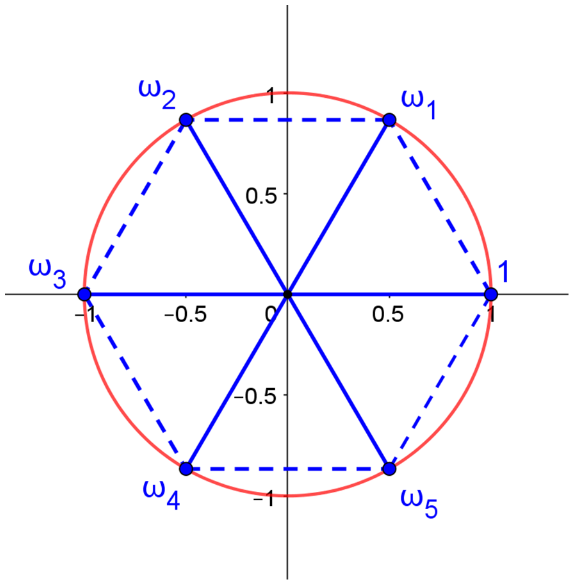
Again we see that the fourth roots of unity all lie on a unit circle, with the roots now forming the vertices of a square:



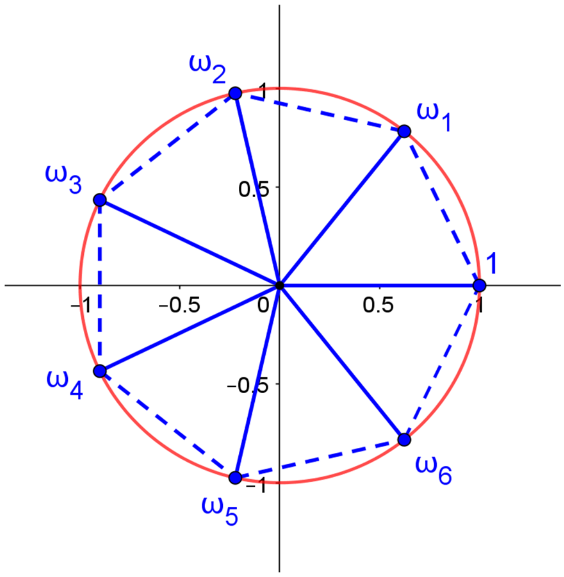
Similar analysis on $x^5 = 1$, $x^6 = 1$, etc. will produce roots which also lie on a unit circle and which form the vertices of a pentagon, hexagon, etc. respectively. It is left as an exercise for you to find the roots and illustrate these on an Argand diagram. The Argand diagram below illustrates the roots of $x^5 = 1$, $x^6 = 1$, $x^7 = 1$, and $x^8 = 1$. It is left as an exercise for you to identify the relevant roots in the diagrams below.



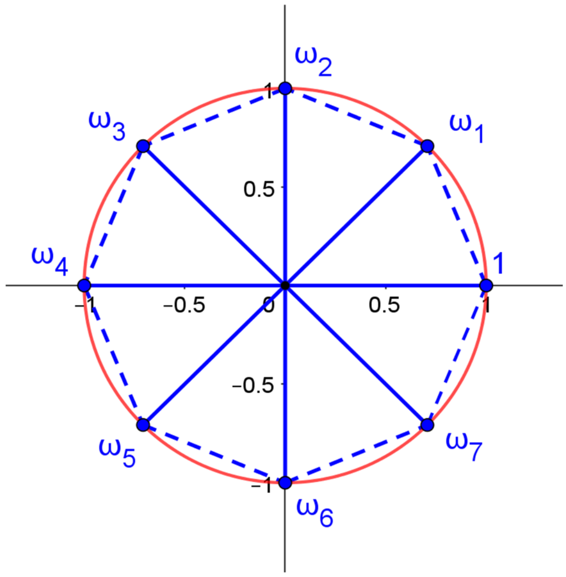
The roots of $x^5 = 1$



The roots of $x^6 = 1$



The roots of $x^7 = 1$



The roots of $x^8 = 1$

In order to find the equation of the roots of unity in the general case of $z^n = 1$, where $n \in \mathbb{N}$, we use the polar form of z and DeMoivre's theorem: let $z = \cos\theta + i\sin\theta$. Then $1 = \cos 0 + i\sin 0$, hence

$$\begin{aligned} z &= (\cos 0 + i\sin 0)^n\,, \\ &= (\cos 2k\pi + i\sin 2k\pi)^n\,. \end{aligned}$$

Therefore the k roots of z are given by

$$z_k = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right), \tag{47}$$

for $k = 0, 1, 2, 3, \ldots, n-1$. Knowing any of the above equations for the sums of roots allows us to simplify expressions involving roots of unity.

Furthermore, if $z^n = m$ for some integer m, then $z^n = m \times 1$ implying $z = \sqrt[n]{m} \times 1^{1/n} = \sqrt[n]{m}\cos(2k\pi/n) + i\sin(2k\pi/n)$. For example, if $z^5 = 13$ then we have $z = \sqrt[5]{13}\cos(2k\pi/n) + i\sin(2k\pi/n)$ where $k = 0, 1, 2, 3, 4$. This gives the following separate roots:

- For $k = 0$, $z_0 = \sqrt[5]{13}\cos(0) + i\sin(0) = \sqrt[5]{13}$;
- For $k = 1$, $z_1 = \sqrt[5]{13}\cos(2\pi/5) + i\sin(2\pi/5)$;
- For $k = 2$, $z_2 = \sqrt[5]{13}\cos(4\pi/5) + i\sin(4\pi/5)$;
- For $k = 3$, $z_3 = \sqrt[5]{13}\cos(6\pi/5) + i\sin(6\pi/5) = \sqrt[5]{13}\cos(4\pi/5) - i\sin(4\pi/5)$;
- For $k = 4$, $z_4 = \sqrt[5]{13}\cos(8\pi/5) + i\sin(8\pi/5) = \sqrt[5]{13}\cos(2\pi/5) - i\sin(2\pi/5)$;

Returning to $z^n = 1$ we can ask, What positive integers n give only real solutions to this equation? Well, we know from (4747) that z_k will be real only if $\sin(2k\pi/n) = 0$. Hence we have

$$\frac{2k\pi}{n} = m\pi \, ,$$

for $m = 0, \pm 1, \pm 2, \ldots$ Hence we need $n = 2k/m$. Then

- if $m = 1$, $\sin(2k\pi/n) = \sin(2k\pi/(2k)) = \sin 2\pi/2 = 0$ for all k;
- if $m = 2$, $\sin(2k\pi/n) = \sin(2k\pi/(k/2)) = \sin 4\pi/2 = 0$ for all k;
- if $m = 3, 4, 5, \ldots$, $\sin(2k\pi/n) = \sin(2k\pi/(2k/m)) = 0 = \sin 2m\pi/2 = 0$ for all k;

etc. hence $n = 2k/m$ guarantees real solutions to $z^n = 1$.

The above example was based on the study of roots of unity. Can we generalise the above to apply to any complex number $z = x + iy$? Yes. In this case we are looking for the condition on n such that z only has real solution. Hence

$$z = x + iy = r(\cos\theta + i\sin\theta) \, ,$$

where $r = |z|$ and $\theta = \arg(z)$. Then

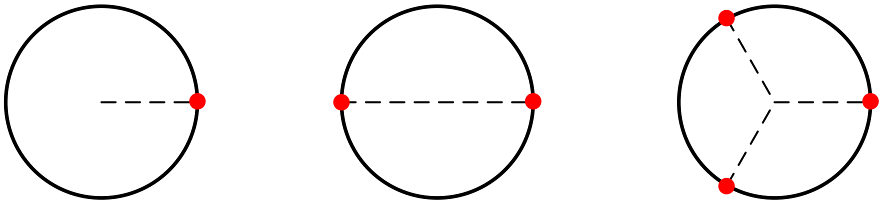
$$z^{1/n} = r^{1/n}(\cos\theta + i\sin\theta)^{1/n} = r^{1/n}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right).$$

Here we need $\sin(\theta/n + 2k\pi/n) = 0$ for there to be only real solution to z, i.e.

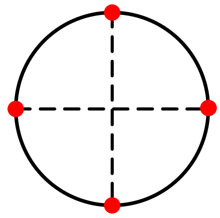
$$n = \frac{\theta + 2k\pi}{m\pi} \, ,$$

for $m = 0, \pm 1, \pm 2, \ldots$

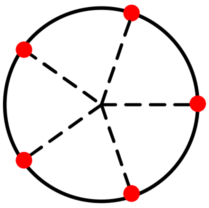
Below is a diagam which illustrates the first nine roots of unity:



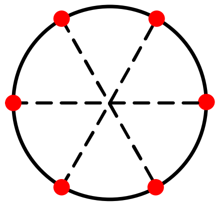
first root *square roots* *cube roots*



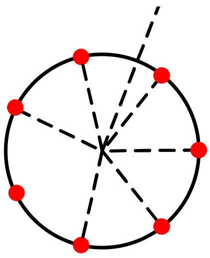
fourth roots



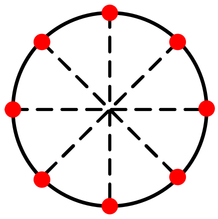
fifth roots



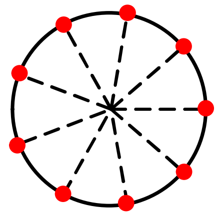
sixth roots



seventh roots



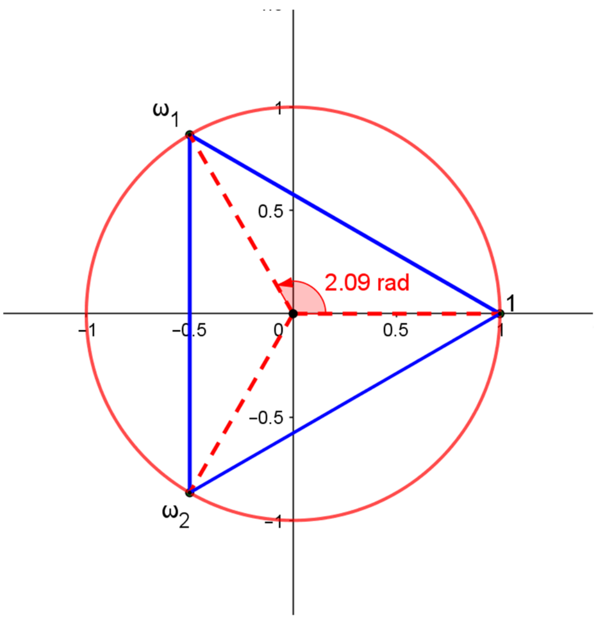
eighth roots



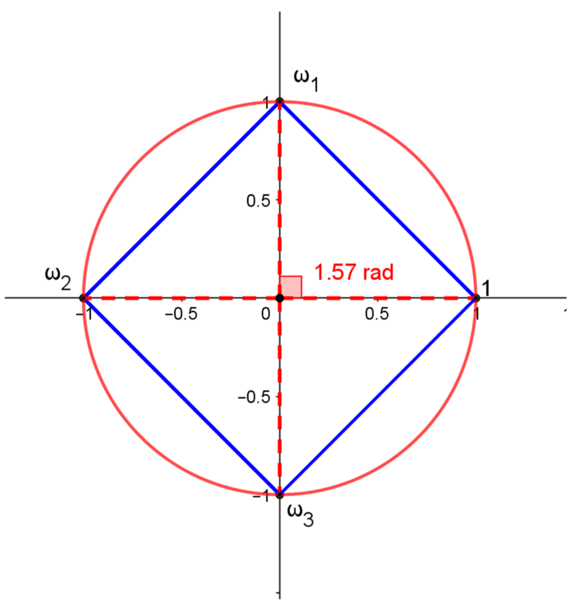
nineth roots

1.15.2 Finding π from a study of roots of unity

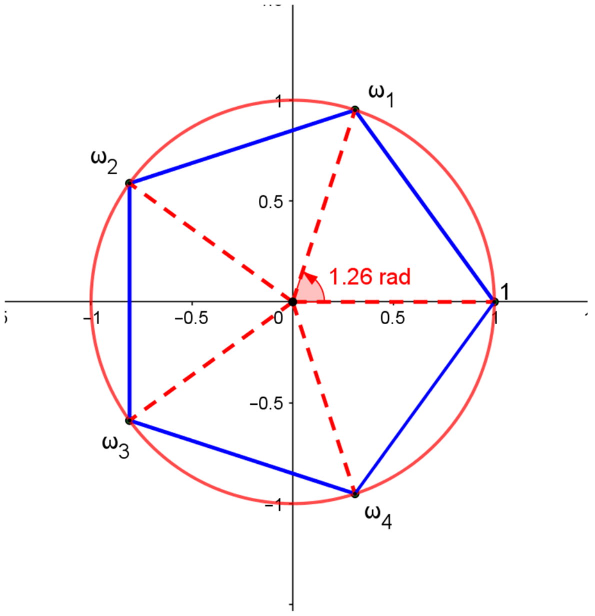
The following is taken from "Roots of unity revisited", Brian Denton, *Mathematical Gazette*, Vol 64, issue 447 (Mar 1985), pp17-20. Consider the polygons generated by the roots of unity $z^3 = 1$, $z^4 = 1$, $z^5 = 1$ and $z^6 = 1$ as illsutrated below:



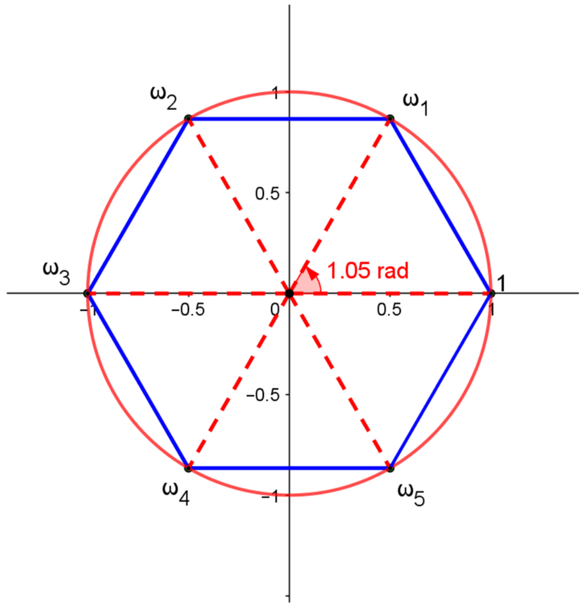
$z^3 = 1$



$z^4 = 1$



$z^5 = 1$



$z^6 = 1$

A question we can ask is, What happens to the area of the polygon as the number of sides n of the polygon approaches infinity?

Another way of putting it is, What happens to the area of the polygon as the number n of triangles that form the polygon approaches infinity? Our aim will be to see if the sequence of values of these areas converges to any particular number.

To start with we need to find a expression for the area of a polygon, this expression being in a form suitable to taking the limit as $n \to \infty$. We do this by considering one trinagle of the polygon. Here we see that each triangle within any polygon is isosceles, with internal angle $\alpha = 2\pi/n$.

So, any one triangle can be generally represented as illustrated below, where b is the base length of the triangle, and h is the perpendicular height to the base of the triangle.

Then the area A_i of a general triangle is given by ½$b \times h$.
Our aim is to find an expression for the area in terms of trig functions. Hence

$$\sin\frac{\alpha}{2} = \frac{b}{2} \qquad \text{implying} \qquad b = 2\sin\frac{\alpha}{2}$$

and

$$h = \cos\frac{\alpha}{2}.$$

So, with $\alpha = 2\pi/n$, we have

$$\begin{aligned} A_i &= \frac{1}{2}\left(2\sin\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2}\right), \\ &= \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right), \\ &= \frac{1}{2}\sin\frac{2\pi}{n}. \end{aligned}$$

Since there are n triangles in a polygon the totlal area of the polygon is

$$A = nA_i = \frac{n}{2}\sin\frac{2\pi}{n}.$$

This gives us the following areas for the polygons formed by the respective roots of unity:

Root of unity equation: $z^n = 1$	**Area**
$z^3 = 1$	$3\sqrt{3}/4 \approx 1.299$
$z^4 = 1$	2
$z^5 = 1$	$\frac{5}{8}\sqrt{2\sqrt{5}+10} \approx 2.378$
$z^6 = 1$	$3\sqrt{3}/2 \approx 2.598$

As n increases we obtain the following sequence of values for the area of the polygon

n	10	...	30	...	50	...	100	...	150	...	1643
Area	2.93893	...	3.11868	...	3.13333	...	3.13953	...	3.14067	...	3.14159

So it looks as if the area of the polygon approaches the value π as $n \to \infty$. This should make sense since, as n increases, so the perimeter of the polygon approaches the perimeter of the circle. The area of a circle of radius 1 is π, so the area of the polygon as $n \to \infty$ should also be π. And we can show this as follows:

$$A = \frac{n}{2}\sin\frac{2\pi}{n} = \frac{\sin(2\pi/n)}{2/n} = \pi\frac{\sin(2\pi/n)}{2\pi/n}.$$

Taking the limit of the LHS as $n \to \infty$ we obtain

$$A = \pi \lim_{n\to\infty} \frac{\sin(2\pi/n)}{2\pi/n} = \pi\,.$$

1.15.3 Sums of roots of unity

Returning to the cube roots of unity, we have roots of 1, and $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Now let $\omega_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\omega_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. By standard algebra we know that, given roots $\alpha_1, \alpha_2, \dots, \alpha_n$, the sum of roots of a real polynomial $ax^n + bx^{n-1} \dots + a_0 = 0$ is given by $\alpha_1 + \alpha_2 + \cdots + \alpha_n = -b/a$. Hence, for $z^3 - 1 = 0$ we have $1 + \omega_1 + \omega_2 = 0$.

Now notice that $\omega_1^2 = \omega_2$. So, letting $\omega_1 = \omega$ we have

$$1 + \omega + \omega^2 = 0,$$

where $\omega^3 = 1$ and $\omega \neq 1$.

For $z^4 - 1 = 0$ we have roots of ± 1 and $\pm i$. Then letting $\omega_0 = 1, \omega_1 = i, \omega_2 = -1$, and $\omega_3 = -i$ we have the sum of the roots to be $1 + i - 1 - i = 0$. Now notice that $\omega_1^2 = -1$ and $\omega_1^3 = -i = \omega_2$, so, letting $\omega_1 = \omega$ we have

$$1 + \omega + \omega^2 + \omega^3 = 0\,,$$

where $\omega^4 = 1$ and $\omega \neq 1$.

<u>*Exercise:*</u> Show that, for $x^5 - 1 = 0$ the sum of the roots are $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$ where $\omega^5 = 1$ and $\omega \neq 1$.

A pattern seems to be developing, and it is true in general that if $z^n - 1 = 0$, for $n \in \mathbb{N}$, then the sum of the roots can be expressed as

$$1 + \omega + \omega^2 + \omega^3 + \cdots + \omega^{n-1} = 0\,,$$

where $\omega^n = 1$ and $\omega \neq 1$. To show this note that $\omega^n = 1 \Longrightarrow \omega^n - 1 = 0$. Hence

$$\omega^n - 1 = (\omega - 1)(\omega^{n-1} + \omega^{n-2} + \cdots + \omega^3 + \omega^2 + \omega + 1) = 0\,.$$

Either $\omega - 1 = 0 \Longrightarrow \omega = 1$, or $\omega^{n-1} + \omega^{n-2} + \cdots + \omega^3 + \omega^2 + \omega + 1 = 0$ where, for this factor, $\omega \neq 1$.

In general we therefore have the following for $z^n = 1$:

n	**Sum of roots**
2	$1 + \omega = 0$
3	$1 + \omega + \omega^2 = 0$
4	$1 + \omega + \omega^2 + \omega^3 = 0$
5	$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$
6	$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 = 0$
7	$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$

etc., with $\omega^n = 1$ for $n = 2, 3, 4, 5, 6, 7$ respectively.

1.15.4 The cyclic nature of roots of unity

The roots of unity have what can be called a *cyclic nature*. In other words, the values of the roots of unity repeat every ω^n. For example, if $z^3 = 1$ we know

$$1 + \omega + \omega^2 = 0\,, \qquad \text{and} \qquad \omega^3 = 1\,.$$
$$(1) \qquad\qquad\qquad\qquad (2)$$

Notice that the root ω^3 in (2) already appears in (1). Multiplying ω^3 by ω, ω^2, and ω^3 we have $\omega^4 = \omega.\omega^3 = \omega$ and $\omega^5 = \omega^2.\omega^3 = \omega^2$ and $\omega^6 = \omega^3.\omega^3 = 1$. Continually multiplying by ω, ω^2 and ω^3 will produce repetition of ω, ω^2 and $\omega^3 = 1$. So we have

$$1 + \omega + \omega^2 = 0\,,$$

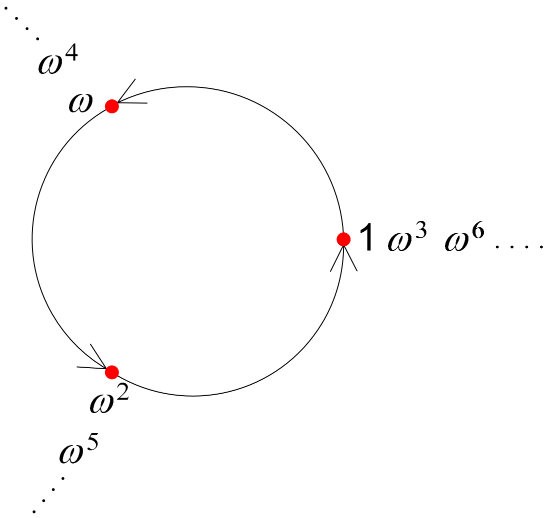
$$\omega^3 + \omega^4 + \omega^5 = 0\,,$$

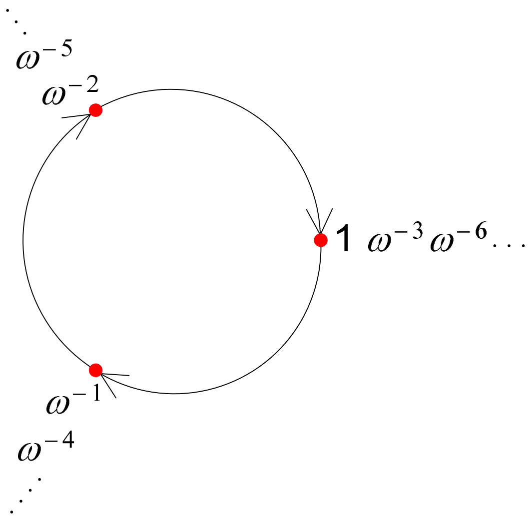
$$\omega^6 + \omega^7 + \omega^8 = 0\,,$$

etc. Similarly we have

$$\omega^{-3} + \omega^{-2} + \omega^{-1} = 0\,,$$

since we can factorise ω^{-3} from this to get $\omega^{-3}(1 + \omega + \omega^2) = w^{-3}(0) = 0$. This situation is represented by the diagrams below.





In general this means that, given $z^n = 1$ is satisfied by the n distinct roots $1, \omega, \omega^2, \ldots, \omega^{n-1}$, so the equation is also satisfied by $\omega^n, \omega^{n+1}, \ldots, \omega^{2n-1}$ and $\omega^{2n}, \omega^{2n+1}, \ldots, \omega^{3n-1}$, etc.

In tabular form we have the following multiplication table for the cube roots of unity and the fourth roots of unity. Similar tables can be developed for higher roots of unity.

Multiplication table for $z^3 = 1$

$\times$	$\boldsymbol{\omega^{-2}}$	$\boldsymbol{\omega^{-1}}$	$\mathbf{1}$
$\boldsymbol{\omega^{-2}}$	ω^2	1	ω
$\boldsymbol{\omega^{-1}}$	1	ω	ω^2
$\mathbf{1}$	ω	ω^2	1

$\times$	$\mathbf{1}$	$\boldsymbol{\omega}$	$\boldsymbol{\omega^2}$
$\mathbf{1}$	1	ω	ω^2
$\boldsymbol{\omega}$	ω	ω^2	1
$\boldsymbol{\omega^2}$	ω^2	1	ω

Multiplication table for $z^4 = 1$

$\times$	$\boldsymbol{\omega^{-3}}$	$\boldsymbol{\omega^{-2}}$	$\boldsymbol{\omega^{-1}}$	$\mathbf{1}$
$\boldsymbol{\omega^{-3}}$	ω^2	ω^4	1	ω
$\boldsymbol{\omega^{-2}}$	ω^4	1	ω	ω^2
$\boldsymbol{\omega^{-1}}$	1	ω	ω^2	ω^3
$\mathbf{1}$	ω	ω^2	ω^3	1

$\times$	$\mathbf{1}$	$\boldsymbol{\omega}$	$\boldsymbol{\omega^2}$	$\boldsymbol{\omega^3}$
$\mathbf{1}$	1	ω	ω^2	ω^3
$\boldsymbol{\omega}$	ω	ω^2	ω^3	1
$\boldsymbol{\omega^2}$	ω^2	ω^3	1	ω
$\boldsymbol{\omega^3}$	ω^3	1	ω	ω^2

Example 1: Suppose that ω is a root of unity of $z^n = 1$. We can show that $|\omega| = 1$ as follows: since ω is a root of unity we have $\omega^n = 1$. Let us take the modulus: $|\omega^n| = |1| = 1$. By the property of modulus $|\omega^n| = |\omega|^n$ we obtain $|\omega|^n = 1$. Hence $|\omega| = 1^{1/n} = 1$.

Example 2: Suppose that ω is a root of unity of $z^n = 1$. We can show that $\omega^* = \omega^{-1}$ as follows: since $\omega^{-1} = 1/\omega$ we have

$$\omega^* = \frac{1}{\omega} = \frac{1}{\omega}.\frac{\omega^*}{\omega^*} = \frac{\omega^*}{|\omega|^2},$$

where we have use the general property $z.z^* = |z|^2$. Therefore, $\omega^* = \omega^*/|\omega|^2$ implies $|\omega|^2 = 1$ which we know is true. Hence $\omega^* = \omega^{-1}$, or $\omega^*\omega = 1$.

Another way to show this is as follows: $\omega^* = \omega^{-1}$ implies $\omega^*\omega = 1$. Taking the modulus of both sides we obtain $|\omega^*\omega| = |1| \Longrightarrow |\omega^*||\omega| = 1 \Longrightarrow |\omega^*| = 1$, since $|\omega| = 1$ by example 1.

Now, you might think that we also know that $|\omega^*| = 1$, hence we have $1 = 1$ as our final solution, which is obviously true. Hence we have answered the question. But we can't do this. We actually have to show $|\omega^*| = 1$ formally by using whatever prior properties of complex numbers we know. This we can do by taking the conjugate of both sides.

Hence $|\omega^*|^* = 1^* = 1$. Again by the property that $|z^*|^* = z$ we have $|\omega^*|^* = |\omega| = 1$ which we know is true. Hence we have shown that $\omega^* = \omega^{-1}$.

Example 3: If ω is a 3rd root of unity, where $\omega \neq 1$, then we can simplify $(\omega^2 + 1)(3\omega^2 + 2\omega)$ as follows: we know that i) $1 + \omega + \omega^2 = 0$, and ii) $\omega^3 = 1$. Therefore, by i) $-\omega = \omega^2 + 1$ we have

$$(\omega^2 + 1)(3\omega^2 + 2\omega) = (-\omega)(3\omega^2 + 2\omega).$$

Splitting $3\omega^2 + 2\omega$ into combination of i) we have

$$(\omega^2 + 1)(3\omega^2 + 2\omega) = (-\omega)(\omega^2 + \omega + \omega^2 + \omega + \omega^2).$$

Since $\omega^2 + \omega = -1$ we obtain

$$\begin{aligned}(\omega^2 + 1)(3\omega^2 + 2\omega) &= (-\omega)(-2 + \omega^2), \\ &= 2\omega - \omega^3, \\ &= 2\omega - 1,\end{aligned}$$

since $\omega^3 = 1$. Alternatively, we could have expanded the original expression to get

$$\begin{aligned}(\omega^2 + 1)(3\omega^2 + 2\omega) &= 3\omega^4 + 2\omega^3 + 3\omega^2 + 2\omega, \\ &= 3\omega + 2\omega^3 + 3\omega^2 + 2\omega,\end{aligned}$$

since $\omega^4 = \omega.\omega^3 = \omega$ by the fact that $\omega^3 = 1$.

Hence

$$\begin{aligned}(\omega^2+1)(3\omega^2+2\omega) &= 2+3\omega^2+5\omega\,, \\ &= 2+3(-1-\omega)+5\omega\,, \\ &= 2\omega-1\,.\end{aligned}$$

Example 4: Again, if ω is a 3rd root of unity, where $\omega \neq 1$, then $(\omega-1)(\omega^{691}+\omega^*)$ can be simplified as follows:

$$\begin{aligned}(\omega-1)(\omega^{691}+\omega^*) &= (\omega-1)(\omega.\omega^{690}+\omega^*)\,, \\ &= (\omega-1)(\omega.(\omega^3)^{230}+\omega^*)\,, \\ &= (\omega-1)(\omega+\omega^{-1})\,,\end{aligned}$$

where, from previous work, we have $\omega^* = \omega^{-1}$. Expanding the RHS of this last expression, and simplifying, we have

$$\begin{aligned}(\omega-1)(\omega^{691}+\omega^*) &= \omega^2+1-\omega-\omega^{-1}\,, \\ &= -\omega-\omega-\omega^{-1}\,, \\ &= -2\omega-\omega^{-1}.\omega^3\,, \\ &= -2\omega-\omega^2\,, \\ &= 1-\omega\,.\end{aligned}$$

Example 5: Again, if ω is a 3rd root of unity, where $\omega \neq 1$, then $(-4\omega^{-8}-4\omega^{-7})(-7\omega^{-6}+2\omega^{-2})$ can be simplified by knowing that ω^3 and positive integer powers of ω^3 equals 1. Therefore $\omega^{-8} = \omega^{-8}.1 = \omega^{-8}(\omega^3)^3 = \omega^{-8}.\omega^9 = \omega$, and similarly $\omega^{-7} = \omega^2$, $\omega^{-6} = \omega^3 = 1$, and $\omega^{-2} = \omega$.

Hence

$$\begin{aligned}(-4\omega^{-8}-4\omega^{-7})(-7\omega^{-6}+2\omega^{-2}) &= (-4\omega-4\omega^2)(-7+2\omega)\,, \\ &= 28\omega+20\omega^2-8\,, \\ &= 8\omega-28\,.\end{aligned}$$

where this last expression was obtained using $\omega^2 = -1-\omega$ in the equation previous to it.

Example 6: Again, if ω is a 3rd root of unity, where $\omega \neq 1$, then $((7\omega^{-4})^* + 8\omega^3)(\omega^8 - (3\omega^{-6})^*)$ can be simplified in the same way as in the previous example, and also by knowing that $\omega^{-1} = \omega^*$. Therefore $(7\omega^{-4})^* = 7\omega^4$ and $(3\omega^{-6})^* = 3\omega^6$. Hence

$$
\begin{aligned}
((7\omega^{-4})^* + 8\omega^3)(\omega^8 - (3\omega^{-6})^*) &= (7\omega^4 + 8\omega^3)(\omega^8 - 3\omega^6)\,, \\
&= (7\omega^3\omega + 8\omega^3)((\omega^3)^2\omega^2 - 3(\omega^3)^2)\,, \\
&= (7\omega + 8)(\omega^2 - 3)\,, \\
&= 7\omega^3 - 21\omega + 8\omega^2 - 24\,, \\
&= 7 - 21\omega + 8(-1 - \omega) - 24\,, \\
&= -25 - 29\omega\,.
\end{aligned}
$$

Example 7: Again, if ω is a 3rd root of unity, where $\omega \neq 1$, then to simplify the expression $(3\omega + 1)/(2\omega - 1)$ we first take the conjugate of the denominator:

$$
\begin{aligned}
\frac{(3\omega + 1)}{(2\omega - 1)} &= \frac{(3\omega + 1)}{(2\omega - 1)} \times \frac{(2\omega^* - 1)}{(2\omega^* - 1)}\,, \\
&= \frac{6\omega\omega^* - 3\omega + 2\omega^* - 1}{4\omega\omega^* - 2\omega - 2\omega^* + 1}\,.
\end{aligned}
$$

Simplifying, and using $\omega^* = 1/\omega$ (see example 2), we obtain

$$
\begin{aligned}
\frac{(3\omega + 1)}{(2\omega - 1)} &= \frac{5 - 3\omega + 2/\omega}{5 - 2\omega - 2/\omega}\,, \\
&= \frac{5 - 3\omega + 2\omega^2}{5 - 2\omega - 2\omega^2}\,, \\
&= \frac{5 - 3\omega + 2(-1 - \omega)}{5 - 2(\omega + \omega^2)}\,, \\
&= \frac{5 - 3\omega - 2 - 2\omega}{5 - 2(-1)}\,, \\
&= \frac{3 - 5\omega}{7}\,.
\end{aligned}
$$

Example 8: The roots of $z^n = 1$, where $n \in \mathbb{N}$, can be found easily in polar form as follows:

$$\begin{aligned} z &= 1^{1/n}, \\ &= (\cos 0 + i \sin 0)^{1/n}, \\ &= (\cos 2k\pi + i \sin 2k\pi)^{1/n}, \end{aligned}$$

implying

$$z_k = \cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right).$$

Example 9: Suppose we want to find the roots of $x^6 = 1$ which do *not* satisfy $x^2 + x + 1 = 0$, where $x \neq 1$. In order to use complex analysis we first recast the question as a complex numbers questions, i.e. we want to find those roots of $z^6 = 1$ which do not satisfy $\omega^2 + \omega + 1 = 0$, where $\omega \neq 1$. Then, by equation (4747) of section 1.15.1 we have

$$z_k = \cos\left(\frac{2k\pi}{6}\right) + i \sin\left(\frac{2k\pi}{6}\right),$$

for $k = 0, 1, 2, 3, 4, 5$. Hence

- For $k = 0$, $z_0 = \cos(0) + i \sin(0) = 1$;
- For $k = 1$, $z_1 = \cos(2\pi/6) + i \sin(2\pi/6)$;
- For $k = 2$, $z_2 = \cos(4\pi/6) + i \sin(4\pi/6)$;
- For $k = 3$, $z_3 = \cos(6\pi/6) + i \sin(6\pi/6) = -1$;
- For $k = 4$, $z_4 = \cos(8\pi/6) + i \sin(8\pi/6) = \cos(4\pi/6) - i \sin(4\pi/6)$;
- For $k = 5$, $z_5 = \cos(10\pi/6) + i \sin(10\pi/6) = \cos(2\pi/6) - i \sin(2\pi/6)$;

Letting $\omega = z_2$ notice that $z_4 = \omega^2$. These two roots satisfy $1 + \omega + \omega^2 = 0$. However, by testing the remaining roots we see that z_0, z_1, z_3, and z_5 do not satisfy this equation.

Another way of solving this problem, without resorting to complex analysis, is as follows: notice that the equation $x^2 + x + 1 = 0$ represents the sum of the cube roots of unity (where $x \neq 1$). Therefore consider $x^6 = 1$ as a type a cubic equation, specifically $(x^3)^2 - 1^2 = 0$. Hence

$$(x^3)^2 - 1^2 = (x^3 + 1)(x^3 - 1) = 0\,.$$

Therefore $x^3 - 1 = 0$ and/or $x^3 + 1 = 0$, from which we obtain the following:

- for $x^3 - 1 = 0$ we have $x^3 - 1 = (x - 1)(x^2 + x + 1) = 0$, giving us roots $x = 1$ and $x = -\frac{1}{2} \pm \frac{1}{2} i\sqrt{3}$;

- for $x^3 + 1 = 0$ we have $x^3 + 1 = (x + 1)(x^2 - x + 1) = 0$, giving us roots $x = -1$ and $x = \frac{1}{2} \pm \frac{1}{2} i\sqrt{3}$.

Hence, the roots which does not satisfy the sum $1 + x + x^2 = 0$ are $x = 1, -1$, and $\frac{1}{2} \pm \frac{1}{2} i\sqrt{3}$.

Exercise: Which roots of $x^{10} = 1$ make $x^4 + x^3 + x^2 + x + 1 = 0$?

Example 10: If ω is a root of unity of $z^n = 1$ simplify

$$1 + \omega + 2\omega^2 + 3\omega^3 + \cdots + (n-1)\omega^{n-1}.$$

Solution: Let us try a few cases first to see if we can spot a pattern:

$n = 3$: $$1 + \omega + 2\omega^2 = 1 + \omega + \omega^2 + \omega^2 = \omega^2$$

$n = 4$: $$1 + \omega + 2\omega^2 + 3\omega^3 = 1 + \omega + \omega^2 + \omega^3 + \omega^2 + 2\omega^3 = \omega^2 + 2\omega^3$$

$n = 5$: $$\begin{aligned} 1 + \omega + 2\omega^2 + 3\omega^3 + 4\omega^4 = 1 + \omega + \omega^2 + \omega^3 + \omega^4 & \\ +\omega^2 + 2\omega^3 + 3\omega^4 &= \omega^2 + 2\omega^3 + 3\omega^4 \end{aligned}$$

So, in general, it looks like we have

$$1 + \omega + 2\omega^2 + 3\omega^3 + \cdots + (n-1)\omega^{n-1} = \omega^2 + 2\omega^3 + 3\omega^4 + \cdots + (n-2)\omega^{n-1}.$$

It is left as an exercise to proce this by induction.

1.15.5 More complicated examples

Example 1: Consider wanting to solve $(x + 2)^n + x^n = 0$ when $n = 2$. We can do this using standard algebra: $x^2 + 4x + 4 + x^2 = 0$ implying $2x^2 + 4x + 4 = 0$, from which the quadratic formula gives roots $x = -1 \pm i$ via the quadratic formula.

But suppose we now want to solve $(x + 2)^n + x^n = 0$ for higher powers. How are we going to deal with the case of $n = 6$ or $n = 11$ or some other value? Well, the efficient way is to recast the equation as a complex valued equation, which then allows us to use all our complex analysis theory. In the case of $n = 6$ we write $(z + 2)^6 + z^6 = 0$.

Hence

$$\begin{aligned}
\frac{(z+2)^6}{z^6} &= -1\,, \\
\Longrightarrow \qquad \frac{z+2}{z} &= (-1)^{1/6}\,, \\
\Longrightarrow \qquad 1+\frac{2}{z} &= (\cos(\pi+2k\pi)+i\sin(\pi+2k\pi))^{1/6}\,, \\
&= \cos\left(\frac{\pi+2k\pi}{6}\right)+i\sin\left(\frac{\pi+2k\pi}{6}\right),
\end{aligned}$$

for $k = 0, 1, 2, 3, 4, 5$. Evaluating this last equation for each k we obtain

- for $k = 0$ we have $1 + 2/z = \cos(\pi/6) + i\sin(\pi/6)$ implying

  $$\frac{2}{z} = -1 + \frac{\sqrt{3}}{2} + \frac{i}{2}\ ,$$

  from which z can be found (left as an exercise);

- for $k = 1$ we have $1 + 2/z = \cos(3\pi/6) + i\sin(3\pi/6)$ implying

  $$\frac{2}{z} = -1 + i\ ,$$

  from which z can be found (left as an exercise);

- for $k = 2$ we have $1 + 2/z = \cos(5\pi/6) + i\sin(5\pi/6)$ implying

  $$\frac{2}{z} = -1 - \frac{\sqrt{3}}{2} + \frac{i}{2}\ ,$$

  from which z can be found (left as an exercise);

- for $k = 3$ we have $1 + 2/z = \cos(7\pi/6) + i\sin(7\pi/6)$ implying

  $$\frac{2}{z} = -1 - \frac{\sqrt{3}}{2} - \frac{i}{2}\ ,$$

  from which z can be found (left as an exercise);

- for $k = 4$ we have $1 + 2/z = \cos(9\pi/6) + i\sin(9\pi/6)$ implying

  $$\frac{2}{z} = -1 - i\,,$$

  from which z can be found (left as an exercise);

- for $k = 5$ we have $1 + 2/z = \cos(11\pi/6) + i\sin(11\pi/6)$ implying

$$\frac{2}{z} = -1 + \frac{\sqrt{3}}{2} - \frac{i}{2},$$

from which z can be found (left as an exercise).

Example 2: To solve the equation

$$(x-m)^n - x^n = 0\,,$$

where $m, n \in \mathbb{R}$, we would proceed as follows: convert the equation into a complex number equation, i.e. $(z-m)^n - z^n = 0$. Then

$$\begin{aligned} (z-m)^n - z^n &= 0\,, \\ \Rightarrow \qquad \frac{(z-m)^n}{z^n} &= 1\,, \\ \therefore \qquad z - m &= z.1^{1/n}\,, \\ &= z\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right). \end{aligned}$$

Where $k = 0, 1, 2, \dots, n-1$. Hence

$$z\left(1 - \cos\frac{2k\pi}{n} - i\sin\frac{2k\pi}{n}\right) = m\,.$$

Using the trig identity $\cos 2\theta = 1 - 2\sin^2\theta$ we have

$$z\left(2\sin^2\frac{k\pi}{n} - i\sin\frac{2k\pi}{n}\right) = m\,,$$

and using $\sin 2\theta = 2\sin\theta\cos\theta$ we now have

$$\begin{aligned} z\left(2\sin^2\frac{k\pi}{n} - 2i\sin\frac{k\pi}{n}\cos\frac{k\pi}{n}\right) &= m\,, \\ \Rightarrow \qquad 2z.\sin\frac{k\pi}{n}\left(\sin\frac{k\pi}{n} - i\cos\frac{k\pi}{n}\right) &= m\,. \end{aligned}$$

Solving for x means we will have to divide by the bracketed term. This will have the same effect as multiplying both sides by the conjugate of the bracketed term. Doing so, and simplifying, gives us

$$2z.\sin\frac{k\pi}{n} = m\left(\sin\frac{k\pi}{n} + i\cos\frac{k\pi}{n}\right),$$

after which we can now divide through to get

$$z = \frac{m}{2}\left(1 + i\cot\frac{k\pi}{n}\right),$$

for $k = 0, 1, 2, \dots, n-1$. Now things become tricky: if $k \to 0$, $z \to \infty$. This might seem to not be a valid answer. However, in more advanced complex analysis theory $z \to \infty$ is a valid answer. The reason for this is related to something called Riemann sphere and stereographic projection. Since this takes us way beyond the scope of these notes we won't discuss it here. This is just to note that $k = 0$ is a valid parameter to use in the above result.

Exercises: Solve the following:

1) $\dfrac{(1+x)^n}{(1-x)^n} = 1$, Answer: $x = i\tan\left(\frac{k\pi}{n}\right)$, $k = 0, 1, 2, \dots, n-1$

2) $(x+i)^6 + (x-i)^6 = 0$, Answer: $x = \cot\frac{(2k-1)\pi}{12}$, $k = 0, 1, 2, 3, 4, 5$

3) $(1-ix)^n + i(1+ix)^n = 0$, Answer: $x = \tan\frac{(4k+1)\pi}{4n}$, $k = 0, 1, 2, \dots, n-1$

Example 3: Are there any numbers whch satisfy $x^n = x$, apart from 0 and 1? Yes. To see this convert the equation as a complex numbers equation, $z^n = z$. Then $z^n/z = z^{n-1} = 1$ is a roots of unity equation. Hence we have

$$\omega_k = \cos\left(\frac{2\pi k}{n-1}\right) + i\sin\left(\frac{2\pi k}{n-1}\right),$$

for $k = 0, 1, 2, .., n-2$.

Example 4: If ω is a complex eighth root of unity we can show that $\omega + \omega^7$ is real. To do this we will not do any algebra on the relation $1 + \omega + \omega^2 + \cdots + \omega^6 = 0$. Instead we go straight to DeMoivre's theorem. Therefore, $z^8 = 1$ implies $z = 1^{1/8}$. Hence we have

$$z = \cos\left(\frac{2k\pi}{8}\right) + i\sin\left(\frac{2k\pi}{8}\right).$$

So one root is

$$\omega = \cos\left(\frac{2k\pi}{8}\right) + i\sin\left(\frac{2k\pi}{8}\right),$$

and another root is $\omega^7 = \cos(14k\pi/8) + i\sin(14k\pi/8)$ which in principal argument form is

$$\omega^7 = \cos\left(\frac{2k\pi}{8}\right) - i\sin\left(\frac{2k\pi}{8}\right).$$

Hence

$$\omega + \omega^7 = \cos\left(\frac{2k\pi}{8}\right) + i\sin\left(\frac{2k\pi}{8}\right) + \cos\left(\frac{2k\pi}{8}\right) - i\sin\left(\frac{2k\pi}{8}\right),$$

implying

$$\omega + \omega^7 = \cos\left(\frac{2k\pi}{8}\right)$$

which is real.

<u>*Exercise:*</u> What can you say about $\omega^2 + \omega^6$, $\omega^3 + \omega^5$, $2\omega^4$, and $\omega - \omega^7$, $\omega^2 - \omega^6$, $\omega^3 - \omega^5$?

1.15.6 Products of roots of unity

Just as there is a pattern for the sums of roots of unity, so there is now for the product of roots of unity. Both of these patterns are summarised in the table below, where "ROU" stands for "Roots of Unity" and where I have used the "cis" notation for brevity:

ROU equation $z^n = 1$	**Roots of unity: ω_k**	**Sum and product of ROU**
$z^2 = 1$	$\omega_0 = \operatorname{cis} 0$ $\omega_1 = \operatorname{cis} \pi$	Sum: $\omega_0 + \omega_1 = 0$ Product: $\omega_0\omega_1 = -1$
$z^3 = 1$	$\omega_0 = \operatorname{cis} 0$ $\omega_1 = \operatorname{cis}\frac{2\pi}{3}$ $\omega_1 = \operatorname{cis}\frac{-2\pi}{3}$	Sum: $\omega_0 + \omega_1 + \omega_2 = 0$ Product: $\omega_0\omega_1\omega_2 = 1$
$z^4 = 1$	$\omega_0 = \operatorname{cis} 0$ $\omega_1 = \operatorname{cis}\frac{\pi}{2}$ $\omega_2 = \operatorname{cis} \pi$ $\omega_3 = \cos\frac{-\pi}{2}$	Sum: $\omega_0 + \omega_1 + \omega_2 + \omega_3 = 0$ Product: $\omega_0\omega_1\omega_2\omega_3 = -1$

$z^5 = 1$	$\omega_0 = \text{cis}\,0$ $\omega_1 = \text{cis}\,\frac{2\pi}{5}$ $\omega_2 = \text{cis}\,\frac{4\pi}{5}$ $\omega_3 = \cos\frac{-4\pi}{5}$ $\omega_4 = \cos\frac{-2\pi}{5}$	Sum: $\omega_0 + \omega_1 + \omega_2 + \omega_3 + \omega_4 = 0$ Product: $\omega_0\omega_1\omega_2\omega_3\omega_4 = 1$
$z^6 = 1$	$\omega_0 = \text{cis}\,0$ $\omega_1 = \text{cis}\,\frac{\pi}{3}$ $\omega_2 = \text{cis}\,\frac{2\pi}{3}$ $\omega_3 = \cos\pi$ $\omega_4 = \cos\frac{-2\pi}{3}$ $\omega_5 = \text{cis}\,\frac{-\pi}{3}$	Sum: $\omega_0 + \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 = 0$ Product: $\omega_0\omega_1\omega_2\omega_3\omega_4\omega_5 = -1$

So it is that the product of the roots of unity equal 1 when *n* is odd, and −1 when *n* is even. Using the symbols Π (capital pi) to denote product (i.e. if n is a positive integer then $\prod_{k=1}^{n} k = 1 \times 2 \times 3 \times \ldots \times n$) we can summarise the right hand column in the table above as

$$\sum_{k=0}^{n-1} \omega_k = 0 \qquad \text{and} \qquad \prod_{k=0}^{n-1} \omega_k = \begin{cases} 1 & \text{when } n \text{ is odd} \\ -1 & \text{when } n \text{ is even} \end{cases}$$

1.15.7 Primitive roots of unity

At the end of section 1.15.1 we saw a diagram of the distribution of the first nine roots of unity. Closer inspection of each of these roots will show that some roots of unity occur as roots of lower order equations.

For example,

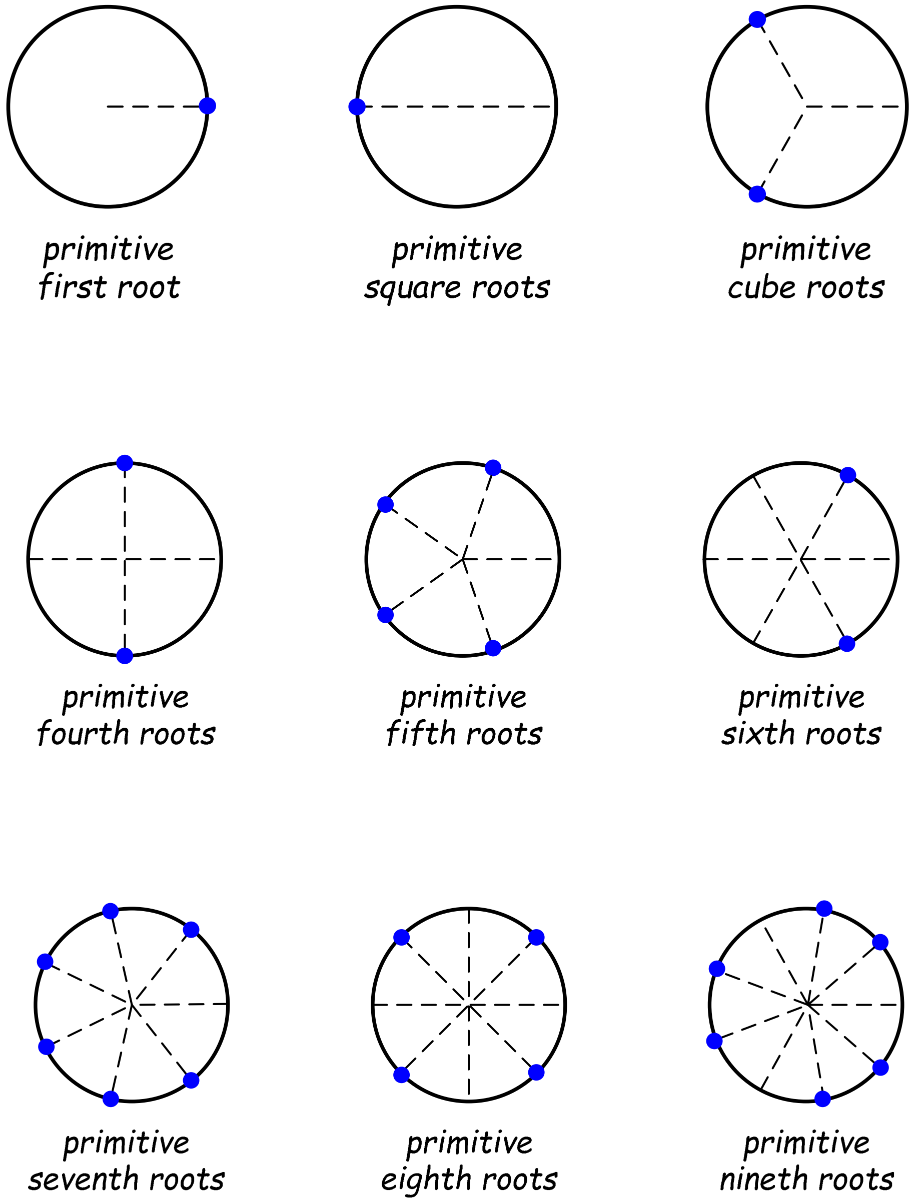
- $\omega = 1$ is a root not only for $z = 1$ but also of $z^n = 1$ for all integer n. Hence $\omega = 1$ is called a *primitive root of unity*;
- $\omega = -1$ is a root not only of $z^2 = 1$ but also of $z^4 = 1$, $z^6 = 1$, $z^8 = 1$, and certain other unit equations. Hence $\omega = -1$ is a primitive root of unity;
- $\omega = -1/2 \pm i\sqrt{3}/2$ is a root not only of $z^3 = 1$ but also of $z^6 = 1$, $z^9 = 1$, and certain other unit equations. Hence $\omega = -1$ is a primitive root of unity;
- $\omega = \pm i$ is a root not only of $z^4 = 1$ but also of $z^8 = 1$ and certain other unit equations. Hence $\omega = \pm i$ is a primitive root of unity;

etc.

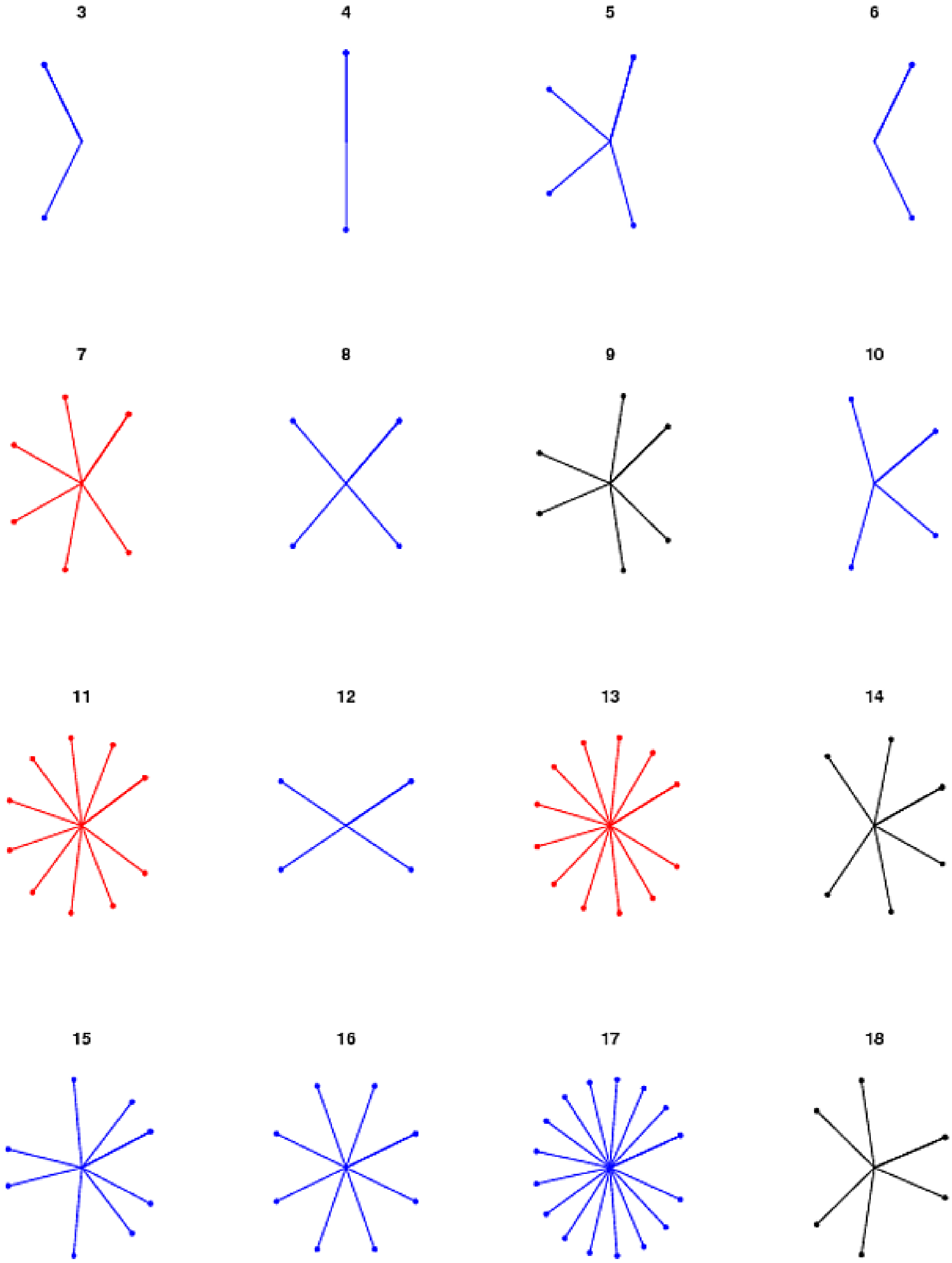
A list of primitive roots of unity for $z^n = 1$ from $n = 1$ to $n = 9$ is shown in the table below, where the primitive roots are highlighted in bold. In this table $\operatorname{cis}\theta$ represents $\cos\theta + i\sin\theta$:

k:	0	1	2	3	4	5	6	7	8
$z^1 = 1$	$\mathbf{cis\,0}$								
$z^2 = 1$	$\operatorname{cis} 0$	$\mathbf{cis}\,\boldsymbol{\pi}$							
$z^3 = 1$	$\operatorname{cis} 0$	$\mathbf{cis}\,\boldsymbol{\frac{2\pi}{3}}$	$\mathbf{cis}\,\boldsymbol{\frac{4\pi}{3}}$						
$z^4 = 1$	$\operatorname{cis} 0$	$\mathbf{cis}\,\boldsymbol{\frac{2\pi}{4}}$	$\operatorname{cis}\frac{4\pi}{4}$	$\mathbf{cis}\,\boldsymbol{\frac{6\pi}{4}}$					
$z^5 = 1$	$\operatorname{cis} 0$	$\mathbf{cis}\,\boldsymbol{\frac{2\pi}{5}}$	$\mathbf{cis}\,\boldsymbol{\frac{4\pi}{5}}$	$\mathbf{cis}\,\boldsymbol{\frac{6\pi}{5}}$	$\mathbf{cis}\,\boldsymbol{\frac{8\pi}{5}}$				
$z^6 = 1$	$\operatorname{cis} 0$	$\mathbf{cis}\,\boldsymbol{\frac{2\pi}{6}}$	$\operatorname{cis}\frac{4\pi}{6}$	$\operatorname{cis}\frac{6\pi}{6}$	$\operatorname{cis}\frac{8\pi}{6}$	$\mathbf{cis}\,\boldsymbol{\frac{10\pi}{6}}$			
$z^7 = 1$	$\operatorname{cis} 0$	$\mathbf{cis}\,\boldsymbol{\frac{2\pi}{7}}$	$\mathbf{cis}\,\boldsymbol{\frac{4\pi}{7}}$	$\mathbf{cis}\,\boldsymbol{\frac{6\pi}{7}}$	$\mathbf{cis}\,\boldsymbol{\frac{8\pi}{7}}$	$\mathbf{cis}\,\boldsymbol{\frac{10\pi}{7}}$	$\mathbf{cis}\,\boldsymbol{\frac{12\pi}{7}}$		
$z^8 = 1$	$\operatorname{cis} 0$	$\mathbf{cis}\,\boldsymbol{\frac{2\pi}{8}}$	$\operatorname{cis}\frac{4\pi}{8}$	$\mathbf{cis}\,\boldsymbol{\frac{6\pi}{8}}$	$\operatorname{cis}\frac{8\pi}{8}$	$\mathbf{cis}\,\boldsymbol{\frac{10\pi}{8}}$	$\operatorname{cis}\frac{12\pi}{8}$	$\mathbf{cis}\,\boldsymbol{\frac{14\pi}{8}}$	
$z^9 = 1$	$\operatorname{cis} 0$	$\mathbf{cis}\,\boldsymbol{\frac{2\pi}{9}}$	$\mathbf{cis}\,\boldsymbol{\frac{4\pi}{9}}$	$\operatorname{cis}\frac{6\pi}{9}$	$\mathbf{cis}\,\boldsymbol{\frac{8\pi}{9}}$	$\mathbf{cis}\,\boldsymbol{\frac{10\pi}{9}}$	$\operatorname{cis}\frac{12\pi}{9}$	$\mathbf{cis}\,\boldsymbol{\frac{14\pi}{9}}$	$\mathbf{cis}\,\boldsymbol{\frac{16\pi}{9}}$

These roots are illustrated in the diagram below, where only primitive roots have been highlighted as blue dots:



The first fifty roots of unity, taken from https://imgur.com/gallery/YdrNuJ8, can be seen in the diagram below. I can't find the original author of these diagrams, so if you know who created them please let me know.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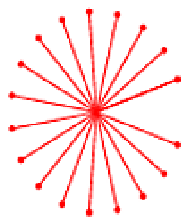

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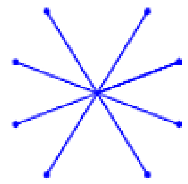

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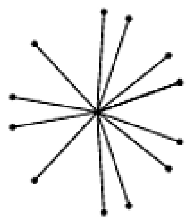

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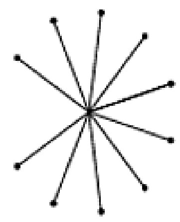

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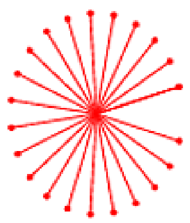

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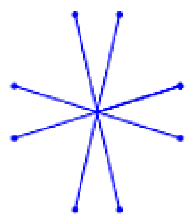

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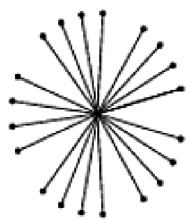

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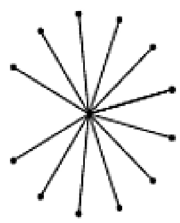

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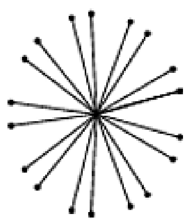

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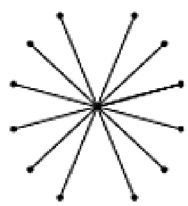

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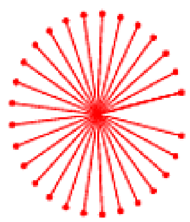

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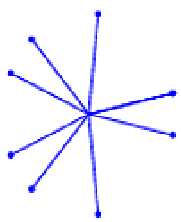

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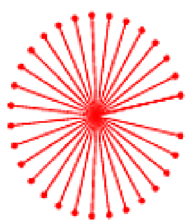

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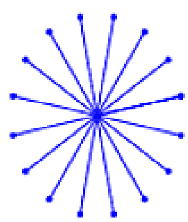

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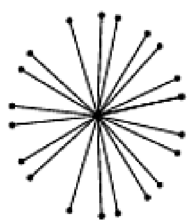

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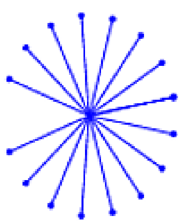

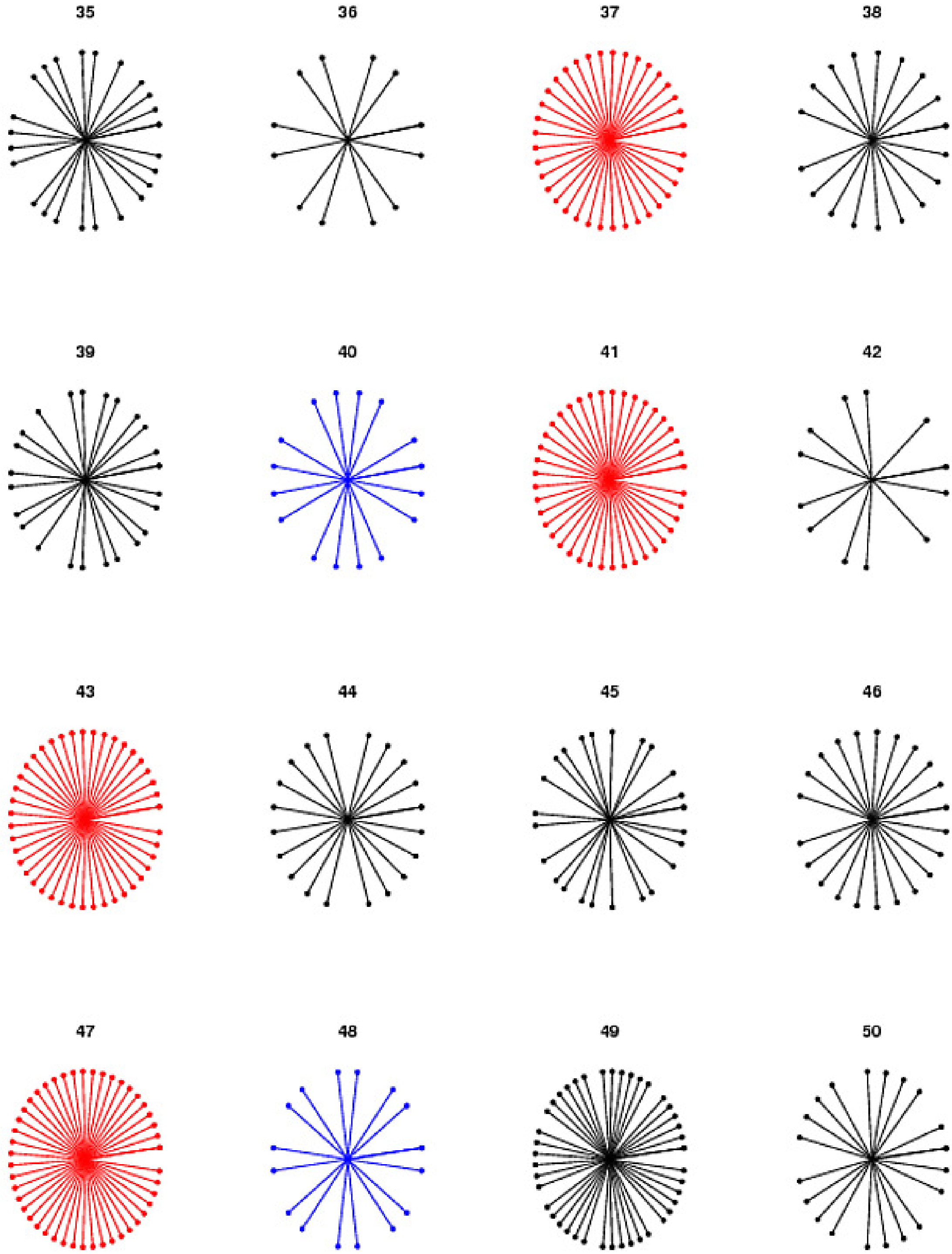
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1.15.8 Certain properties of roots of unity

Here we will go through proving certain properties of roots of unity. In this section let ω_k by a root of unity of $z^n = 1$, where $k, n \in \mathbb{N}$.

<u>*Property 1*</u>: Consider the example of $z^5 = 1$. Then $\omega_k = \cos(2k\pi/5) + i\sin(2k\pi/5)$. So

$$\omega_1 = \cos(2\pi/5) + i\sin(2\pi/5) \qquad \text{and} \qquad \omega_3 = \cos(6\pi/5) + i\sin(6\pi/5)\,.$$

But also, $\omega_4 = \cos(8\pi/5) + i\sin(8\pi/5)$, so $\omega_1\omega_3 = \omega_{1+3} = \omega_4$. It happens to be the case that, in general, we have $\omega_k\omega_j = \omega_{k+j}$.

<u>Proof:</u>

$$\begin{aligned}
\omega_k\omega_j &= \left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right)\left(\cos\left(\frac{2j\pi}{n}\right) + i\sin\left(\frac{2j\pi}{n}\right)\right), \\
&= \cos\left(\frac{2(k+j)\pi}{n}\right) + i\sin\left(\frac{2(k+j)\pi}{n}\right), \\
&= \omega_{k+j}\,.
\end{aligned}$$

■

<u>*Property 2*</u>: Consider again the example of $z^5 = 1$. Then $\omega_1 = \cos(2\pi/5) + i\sin(2\pi/5)$, and $\omega_{-1} = \cos(-2\pi/5) + i\sin(-2\pi/5)$. Now notice that $(\omega_1)^{-1} = (\cos(2\pi/5) + i\sin(2\pi/5))^{-1} = \cos(-2\pi/5) + i\sin(-2\pi/5) = \omega_1$. In general we have $(\omega_k)^{-1} = \omega_{-k}$.

<u>Proof:</u>

$$\begin{aligned}
(\omega_k)^{-1} &= \left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right)^{-1}, \\
&= \cos\left(-\frac{2k\pi}{n}\right) + i\sin\left(-\frac{2k\pi}{n}\right), \\
&= \omega_{-k}\,.
\end{aligned}$$

■

Property 3: Consider again the example of $z^5 = 1$. Then $\omega_1 = \cos(2\pi/5) + i\sin(2\pi/5)$, and $\omega_2 = \cos(4\pi/5) + i\sin(4\pi/5)$. Now consider $(\omega_1)^7$:

$$\begin{aligned}
(\omega_1)^7 &= \left(\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\right)^7, \\
&= \cos\left(\frac{14\pi}{5}\right) + i\sin\left(\frac{14\pi}{5}\right), \\
&= \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right), \\
&= \omega_2 .
\end{aligned}$$

In other words, powering ω_1 gives us another root of unity.

As another example,

$$\begin{aligned}
(\omega_3)^3 &= \left(\cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right)\right)^3, \\
&= \cos\left(\frac{18\pi}{5}\right) + i\sin\left(\frac{18\pi}{5}\right), \\
&= \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right), \\
&= \omega_4 .
\end{aligned}$$

To see why this is so notice that

$$\begin{aligned}
\cos\left(\frac{18\pi}{5}\right) + i\sin\left(\frac{18\pi}{5}\right) &= \cos\left(\frac{(10+8)\pi}{5}\right) + i\sin\left(\frac{(10+8)\pi}{5}\right), \qquad (*) \\
&= \cos\left(\frac{10\pi}{5} + \frac{8\pi}{5}\right) + i\sin\left(\frac{10\pi}{5} + \frac{8\pi}{5}\right), \\
&= \cos\left(2\pi + \frac{8\pi}{5}\right) + i\sin\left(2\pi + \frac{8\pi}{5}\right), \qquad (**) \\
&= \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right), \\
&= \omega_4 .
\end{aligned}$$

The aim is therefore to perform the arithmetic split shown in (*) such that we obtain 2π (or a multiple of 2π) as a term inside the brackets as shown in (**).

In general it is the case that raising any root of unity to an integer power will result in another root of unity. In other words, if ω_k is a root of unity of $z^n = 1$, then $(w_k)^p$ is also a root of unity of $z^n = 1$, where $p \in \mathbb{N}$.

Proof:

$$\begin{aligned} (\omega_k)^p &= \left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right)^p, \\ &= \cos\left(\frac{2pk\pi}{n}\right) + i\sin\left(\frac{2pk\pi}{n}\right). \end{aligned}$$

The quick way of describing the effect of this last equation is that $2pk\pi/n$ is an integer multiple of $2k\pi/n$ from which all distinct roots of unity are derived. Hence $(\omega_k)^p$ is also a root of unity. More properly we have

$$\begin{aligned} (\omega_k)^p &= \cos\left(\frac{2pk\pi}{n}\right) + i\sin\left(\frac{2pk\pi}{n}\right), \\ &= \cos\left(\frac{(mn+q)2k\pi}{n}\right) + i\sin\left(\frac{(mn+q)2k\pi}{n}\right), \end{aligned}$$

where m, n, q are positive integers. Transforming p into $mn + q$ is basically an application of the fundamental theorem of arithmetic which says that any number p (odd or even) can be written as a product of two integers plus a remainder. In our case we want to write p in such a way as to contain n in the product. By doing this we then have

$$\begin{aligned} (\omega_k)^p &= \cos\left(\frac{2mnk\pi}{n} + \frac{2qk\pi}{n}\right) + i\sin\left(\frac{2mnk\pi}{n} + \frac{2qk\pi}{n}\right), \\ &= \cos\left(2mk\pi + \frac{2qk\pi}{n}\right) + i\sin\left(2mk\pi + \frac{2qk\pi}{n}\right), \\ &= \cos\left(\frac{2qk\pi}{n}\right) + i\sin\left(\frac{2qk\pi}{n}\right), \\ &= \omega_q\,. \end{aligned}$$

where ω^q is a root of unity for $q = 0, 1, 2, 3 \ldots.$ ■

Exercise: Prove that, if ω_2 is a root of unity, then $(\omega_2)^p = \omega_{2p}$, where $p \in \mathbb{N}$. Can you generalise this to prove that if ω_k is a root of unity, then $(\omega_k)^p = \omega_{kp}$, where $p \in \mathbb{N}$?

Property 4: Consider again the example of $z^5 = 1$. Then $\omega_3 = \cos(6\pi/5) + i\sin(6\pi/5)$. Now notice that

$$\begin{aligned}
\overline{\omega_3} &= \cos\left(\frac{6\pi}{5}\right) - i\sin\left(\frac{6\pi}{5}\right), \\
&= \cos\left(-\frac{6\pi}{5}\right) + i\sin\left(-\frac{6\pi}{5}\right), \\
&= \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right), \\
&= \omega_2 = \omega_{5-3}\,.
\end{aligned}$$

Hence, the conjugate of ω_3 gives another root of unity. In general we have $\overline{\omega_k} = \omega_{n-k}$.

<u>Proof:</u>

$$\begin{aligned}
\overline{\omega_k} &= \cos\left(\frac{2k\pi}{n}\right) - i\sin\left(\frac{2k\pi}{n}\right), \\
&= \cos\left(-\frac{2k\pi}{n}\right) + i\sin\left(-\frac{2k\pi}{n}\right), \\
&= \cos\left(2\pi - \frac{2k\pi}{n}\right) + i\sin\left(2\pi - \frac{2k\pi}{n}\right), \\
&= \cos\left(\frac{2n\pi - 2k\pi}{n}\right) + i\sin\left(\frac{2n\pi - 2k\pi}{n}\right), \\
&= \cos\left(\frac{2(n-k)\pi}{n}\right) + i\sin\left(\frac{2(n-k)\pi}{n}\right), \\
&= \omega_{n-k}\,.
\end{aligned}$$

■

Property 5: Let ω be a cube root of unity. Then, for $n \in \mathbb{N}$,

$$1^n + \omega^n + (\omega^2)^n = \begin{cases} 0 & \text{if } n \text{ is not a multiple of 3;} \\ 3 & \text{if } n \text{ is a multiple of 3;} \end{cases}$$

Proof: We only need to consider three cases: $n = 1, 2, 3$:

- For $n = 1$: $1^1 + \omega^1 + (\omega^2)^1 = 1 + \omega + \omega^2 = 0$, since $n = 1$ is not a multiple of 3;
- For $n = 2$: $1^2 + \omega^2 + (\omega^2)^2 = 1 + \omega^2 + \omega^4 = 1 + \omega^2 + \omega^3\omega = 1 + \omega + \omega^2 = 0$, since $\omega^3 = 1$, and $n = 2$ is not a multiple of 3;
- For $n = 3$: $1^3 + \omega^3 + (\omega^2)^3 = 1 + \omega^3 + (\omega^3)^2 = 1 + 1 + 1 = 3$, since $\omega^3 = 1$, and $n = 3$ is a multiple of 3;

All other values of *n* beyond those three above will only repeat the answers above. A more complete/rigorous proof addressing all values of *n* can be developed by considering the three cases of $n = 1 + 3k$, $n = 2 + 3k$, and $n = 3 + 3k$, for $k = 0, 1, 2, 3, \ldots$ This is left as an exercise.

■

Property 6: It is left as an exercise to prove that, if ω be an n^{th} root of unity,

$$1^m + \omega^m + (\omega^2)^m + \cdots + (\omega^{n-1})^m = \begin{cases} 0 & \text{if } m \text{ is not a multiple of } m; \\ n & \text{if } m \text{ is a multiple of } n; \end{cases}$$

for $m \in \mathbb{N}$.

1.16 On deriving trigonometric identities via complex numbers

We have all seen how proving trig identities can be very laborious using standard trig identities, particularly when the power, or multiple of the angle, is large. However, there is a way of using DeMoivre's theorem which make it considerable easier and shorter to prove trig identities. This is what we shall look into.

1.16.1 Trig identities involving powers of trig functions

Let us now see how we can use DeMoivre's theorem to deal with powers of trig functions. In the last example of the section above we saw that the combination $z + 1/z = 2\cos\theta$ and $z - 1/z = 2i\sin\theta$. By the same process we can show that $z^2 + 1/z^2 = 2\cos 2\theta$ and $z^2 - 1/z^2 = 2i\sin 2\theta$, etc. This can be generalised for any $n \in \mathbb{N}$ as follows: let $z = \cos\theta + i\sin\theta$.

Then

$$\begin{aligned} z^n + \frac{1}{z^n} &= (\cos n\theta + i \sin n\theta) + (\cos n\theta + i \sin n\theta)^{-1} , \\ &= (\cos n\theta + i \sin n\theta) + (\cos n\theta - i \sin n\theta) , \\ &= 2 \cos n\theta , \end{aligned}$$

and

$$\begin{aligned} z^n - \frac{1}{z^n} &= (\cos n\theta + i \sin n\theta) - (\cos n\theta + i. \sin n\theta)^{-1} , \\ &= (\cos n\theta + i \sin n\theta) + (\cos n\theta - i \sin n\theta) , \\ &= 2i. \sin n\theta . \end{aligned}$$

where $n \in \mathbb{N}$. The expressions above of

$$z^n + \frac{1}{z^n} = 2 \cos n\theta \tag{48}$$

$$z^n - \frac{1}{z^n} = 2i. \sin n\theta \tag{49}$$

are important properties of complex numbers (notice that expressions (48) and (49) apply only when $r = |z| = 1$. Otherwise you will have factors of $r^n \pm r^{-n}$ in these expressions).

We are now in a position to be able to find trig identities involving powers of trig functions. For example, to find an identity for $\cos^2 \theta$ we set $n = 1$ equation (48), and then square. Hence

$$(2 \cos \theta)^2 = \left(z + \frac{1}{z}\right)^2 .$$

Expanding both sides as usual we obtain

$$4 \cos^2 \theta = z^2 + 2z \frac{1}{z} + \frac{1}{z^2} .$$

The trick is now to collect terms in the form $z^n + 1/z^n$, viz

$$4 \cos^2 \theta = \left(z^2 + \frac{1}{z^2}\right) + 2z \frac{1}{z} ,$$

this now allowing us to apply (48) to get

$$4 \cos^2 \theta = 2 \cos 2\theta + 2 ,$$

from which we obtain

$$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$$

(i.e. the alternate form of $\cos 2\theta = 2\cos^2\theta - 1$).

The same process can be used to trig identites for any power of cos or sin. E.g. to find the identity for $\sin^3\theta$ we we set $n = 1$ equation (49), and then cube. Hence

$$(2i.\sin\theta)^3 \quad = \quad \left(z - \frac{1}{z}\right)^3,$$

implying

$$8i^3\sin^3\theta \quad = \quad z^3 - z^2\left(\frac{1}{z}\right) + z\left(\frac{1}{z^2}\right) - \frac{1}{z^3}.$$

We now collect terms in the form $z^n \pm 1/z^n$, viz

$$-8i\sin^3\theta = \left(z^3 - \frac{1}{z^3}\right) - \left(z - \frac{1}{z}\right),$$

this now allowing us to apply (49) to get

$$-8i\sin^3\theta = 2i\sin 3\theta - 2i\sin\theta\,,$$

which simplifies to

$$\sin^3\theta = \frac{1}{4}(\sin 3\theta - 2\sin\theta)\,.$$

This process can be repeated to find trig identities for $\cos^4\theta$, $\sin^4\theta$, $\cos^5\theta$, $\sin^5\theta$, etc.

If we want to find a trig identity for $\cos^3 2\theta$ we set $n = 2$ equation (48), and then square. Hence

$$(2\cos 2\theta)^3 = \left(z^2 + \frac{1}{z^2}\right)^3.$$

Expanding both sides as usual we obtain

$$8\cos^3 2\theta = z^6 + 3z^4\frac{1}{z^2} + 3z^2\frac{1}{z^4} + \frac{1}{z^6}.$$

Again we collect terms in the form $z^n + 1/z^n$, i.e.

$$8\cos^3 2\theta = \left(z^6 + \frac{1}{z^6}\right) + 3\left(z^2 + \frac{1}{z^2}\right),$$

this now allowing us to apply (48) to obtain

$$8\cos^3 2\theta = 2\cos 6\theta + 3\cos 2\theta\ .$$

Hence

$$\cos^3 2\theta = \frac{1}{4}\cos 6\theta + \frac{3}{8}\cos 2\theta\ .$$

Notice how quickly and easily we arrive at such a compact result compared to the time and complexity it would take to get this via the use of standard trig identities.

To find trig identities involving multiple angles, such a $2\theta, 3\theta$, etc. we simply apply the above procedure using $n = 2\theta$, $n = 3\theta$, etc., in (48) and/or (49). For example, if we want to find an identity for $\tan^2 3\theta$ we first find $\sin^2 3\theta$ and $\cos^2 3\theta$ as folows:

$$\begin{aligned}
& (2\cos 3\theta)^2 &=& \left(z^3 + \frac{1}{z^3}\right)^2, \\
\Longrightarrow\quad & 4\cos^2 3\theta &=& z^6 + 2z^3\left(\frac{1}{z^3}\right) + \frac{1}{z^6}, \\
& &=& \left(z^6 + \frac{1}{z^6}\right) + 2\,, \\
& &=& 2\cos 6\theta + 2\,, \\
\Longrightarrow\quad & \cos^2 3\theta &=& \frac{1}{2}(\cos 6\theta + 1)\,,
\end{aligned}$$

which is the identity we expect since this is an alternate form of $\cos 2\phi = 2\cos^2\phi - 1$, where $\phi = 3\theta$. Similarly for $\sin^2 3\theta$ we have

$$\begin{aligned}
& (2i\sin 3\theta)^2 &=& \left(z^3 - \frac{1}{z^3}\right)^2, \\
\Longrightarrow\quad & -4\sin^2 3\theta &=& z^6 - 2z^3\left(\frac{1}{z^3}\right) + \frac{1}{z^6}, \\
& &=& \left(z^6 + \frac{1}{z^6}\right) - 2\,, \\
& &=& 2\cos 6\theta - 2\,, \\
\Longrightarrow\quad & \sin^2 3\theta &=& \frac{1}{2}(1 - \cos 6\theta)\,,
\end{aligned}$$

which is again the identity we expect since this is an alternate form of $\cos 2\phi = 1 - 2\sin^2\phi$ where $\phi = 3\theta$.

Now we form $\tan^2 3\theta$ in the usual way:

$$\begin{aligned}\tan^2 3\theta &= \frac{\sin^2 3\theta}{\cos^2 3\theta} \\ &= \frac{1-\cos 6\theta}{\cos 6\theta + 1}.\end{aligned}$$

This expression could be left as it is or it could be simplified. In order to simplify this we could make use of standard trig identites, but because of the angle 6θ we will resist this temptation. Instead, it would be nice if we could find a way of expanding $\cos 6\theta$ and $\sin 6\theta$ some form of using complex analysis. This is what we shall now see in the next section.

1.16.2 Trig identities involving multiples of θ

Let $z = \cos\theta + i\sin\theta$. Let us now evaluate z^2. ByDeMoivre's theorem we have

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta\ .$$

But we can also expand the left hand side of the above equation in the usual way to obtain

$$\cos^2\theta + 2i.\sin\theta\,.\cos\theta - \sin^2\theta = \cos 2\theta + i\sin 2\theta\ .$$

Comparing *Re* and *Im* parts we have

$$\begin{aligned}\cos^2\theta - \sin^2\theta &= \cos 2\theta\ , \\ \text{and} \qquad 2\sin\theta\,.\cos\theta &= \sin 2\theta\ . \qquad (50)\end{aligned}$$

What we have done to $(\cos\theta + i\sin\theta)^2$ bears reiterating: on the one hand we have expanded this expression as usual, and on the other hand we have applied DeMoivre's theorem to it. In doing this we have recovered the standard identities for $\cos 2\theta$ and $\sin 2\theta$.

This is not a coincidence. If we want to find the standard identity for $\cos 3\theta$, we do the same thing. Then, letting $\cos\theta + i\sin\theta \equiv c + is$ we have

$$\begin{aligned}(c+is)^3 &= \cos 3\theta + i\sin 3\theta\ , \qquad \text{(RHS by DeMoivre's theorem)} \\ \Longrightarrow \quad c^3 + 3c^2(is) + 3c(is)^2 + (is)^3 &= \cos 3\theta + i\sin 3\theta\ . \qquad \text{(LHS by standard expansion)}\end{aligned}$$

Comparing *Re* and *Im* parts we have

$$\cos^3\theta - 3\cos\theta\,.\sin^2\theta \quad = \quad \cos 3\theta\ ,$$

and

$$3\cos^2\theta\sin\theta - \sin^3\theta \quad = \quad \sin 3\theta\ .$$

Let us now return to the example involving $\tan^2 3\theta$ in the previous section. We got as far as

$$\tan^2 3\theta = \frac{1-\cos 6\theta}{\cos 6\theta + 1}.$$

We can now use the analysis of this section to simplify the term $\cos 6\theta$ of this expression. Letting $\cos\theta + i\sin\theta \equiv c + is$, DeMoivre's theorem give us

$$(c+is)^6 = \cos 6\theta + i\sin 6\theta\ .$$

Using Pascal's triangle of coefficients on the LHS we obtain

$$c^6 + 6c^5(is) + 15c^4(is)^2 + 20c^3(is)^3 + 15c^2(is)^4 + 6c(is)^5 + (is)^6 = \cos 6\theta + i\sin 6\theta\ .$$

Since we are only interested in the term $\cos 6\theta$ we need only simplify and equate the *Re* part of the above to get

$$c^6 - 15c^4s^2 + 15c^2s^4 - s^6 = \cos 6\theta\ ,$$

Hence

$$\begin{aligned}
\tan^2 3\theta \quad &= \quad \frac{1-\cos 6\theta}{\cos 6\theta + 1}, \\
&= \quad \frac{1-(c^6 - 15c^4s^2 + 15c^2s^4 - s^6)}{(c^6 - 15c^4s^2 + 15c^2s^4 - s^6)+1}, \\
&= \quad \frac{1-c^6 + 15c^4s^2 - 15c^2s^4 + s^6}{c^6 - 15c^4s^2 + 15c^2s^4 - s^6 + 1}.
\end{aligned}$$

Diving by $\cos^6\theta$ and using $1 + \tan^2\theta = \sec^2\theta$ we finally obtain our desired identity

$$\tan^2 3\theta = \frac{18\tan^2\theta - 12\tan^4\theta + 2\tan^6\theta}{2 - 12\tan^2\theta + 18\tan^4\theta}.$$

Notice how much easier it has been to come to this answer despite having to expand a polynomial of degree 6. In general it is considerably easier to derive trig identities via some application of DeMoivre's theorem than it is to do so via the use of standard trig identities.

In fact, there is a saying which goes something like, The quickest way to an answer in the real domain is via the complex domain.

1.16.3 Examples on deriving trig identities

Example 1: What about finding an identity for $\sin^4\theta\,.\cos^3\theta$? Here we use equations (48) and (49). Hence

$$
\begin{aligned}
(\sin\theta)^4(\cos\theta)^3 &= \left(\frac{1}{2i}\right)^4\left(z-\frac{1}{z}\right)^4\left(\frac{1}{2}\right)^3\left(z+\frac{1}{z}\right)^3, \\
&= \left(\frac{1}{16}\right)\left(\frac{1}{8}\right)\left[\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)\right]^3\left(z-\frac{1}{z}\right), \\
&= \left(\frac{1}{128}\right)\left(z^2-\frac{1}{z^2}\right)^3\left(z-\frac{1}{z}\right), \\
&= \frac{1}{128}\left(z^6-3z^2+\frac{3}{z^2}-\frac{1}{z^6}\right)\left(z-\frac{1}{z}\right).
\end{aligned}
$$

Expanding this last equation, and grouping terms according to z^n+1/z^n we obtain

$$(\sin\theta)^4(\cos\theta)^3=\frac{1}{128}\left(z^7+\frac{1}{z^7}\right)-\left(z^5+\frac{1}{z^5}\right)-3\left(z^3+\frac{1}{z^3}\right)+3\left(z+\frac{1}{z}\right).$$

We can now use equations (48) and (49) to convert the RHS back into trig form. This gives us, upon simplifying,

$$\sin^4\theta\,.\cos^3\theta=\frac{1}{164}(\cos^7\theta-\cos^5\theta-3\cos^3\theta+3\cos\theta)\,.$$

Example 2: Suppose we want to show that $\sec\theta\,.\cos 5\theta=1-12\sin^2\theta+16\sin^4\theta$. We can proceed as follows. Letting $\cos\theta+i\sin\theta$ be $c+is$ we have

$$
\begin{aligned}
\cos 5\theta = Re(\cos 5\theta+i\sin 5\theta) &= Re(c+is)^5\,, \\
&= Re(c^5+5ic^4s-10c^3s^2-10ic^2s^3+5cs^4+is^5)\,, \\
&= c^5-10c^3s^2+5cs^4\,.
\end{aligned}
$$

Dividing this last equation by $\cos\theta$ we obtain

$$\sec\theta\,.\cos 5\theta=\cos^4\theta-10\cos^2\theta\sin^2\theta+5\sin^4\theta\,.$$

Using $\cos^2\theta + \sin^2\theta = 1$ we can convert this into the form required:

$$\sec\theta\,.\cos 5\theta = (1-\sin^2\theta)^2 - 10(1-\sin^2\theta)\sin^2\theta + 5\sin^4\theta\,,$$

which simplifes to $\sec\theta\,.\cos 5\theta = 1 - 12\sin^2\theta + 16\sin^4\theta$.

<u>*Example 3:*</u> Suppose we are given $\sin\theta + i\cos\theta$. How do we convert this into the standard polar form of a complex number? Well, we know that $\cos(\pi/2-\theta) = \sin\theta$ and $\sin(\pi/2-\theta) = \cos\theta$, hence

$$\sin\theta + i\cos\theta = \cos(\pi/2-\theta) + i\sin(\pi/2-\theta)\,.$$

From this we can find trig identities based on the expansion of $(\sin\theta + i\cos\theta)^n$ where $n \in \mathbb{N}$. Letting $s + ic$ represent $\sin\theta + i\cos\theta$ we have

$$(s+ic)^n = s^n + \binom{n}{1}s^{n-1}(ic) + \binom{n}{2}s^{n-2}(ic)^2 + \binom{n}{3}s^{n-3}(ic)^3 + \cdots$$

Comnparing *Re* and *Im* parts we see that

- $\cos(\pi/2-\theta) = s^n - \binom{n}{2}s^{n-2}c^2 + \binom{n}{4}s^{n-4}c^4 - \cdots$

and

- $\sin(\pi/2-\theta) = \binom{n}{1}s^{n-1}c - \binom{n}{3}s^{n-3}c^3 + \binom{n}{5}s^{n-5}c^5 - \cdots$

<u>*Example 4*</u> Suppose we want to find an equation which is satisfied by $\tan\theta$ when $\tan 3\theta = 0$. To start with we kno that the tan function comes from sin/cos, so set up $\cos 3\theta + i\sin 3\theta$, use Demoivre's theorem appropriately, do some algebra and then divide the *Im* part by the *Re* part. Therefore, letting $\cos\theta + i\sin\theta \equiv c + is$ we have

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= (c+is)^3\,,\\ &= c^3 + 3c^2(is) + 3c(is)^2 + (is)^3\,,\\ &= c^3 - 3cs^2 + i(3c^2s - s^3)\,.\end{aligned}$$

Hence

$$\tan 3\theta = \frac{3c^2s - s^3}{c^3 - 3cs^2}\,.$$

Now, to get an equation in $\tan\theta$ we divide top and bottom of the right hand side of this last equation by $\cos^3\theta$ to get

$$\tan 3\theta = \frac{3t - t^3}{1 - 3t^2}\,,$$

where $t \equiv \tan\theta$. So, in order for $\tan 3\theta = 0$ we need $3\tan\theta - \tan^3\theta = 0$. As an exercise, try finding similar equations for $\tan\theta$ when $\tan 4\theta = 0$, $\tan 5\theta = 0$, etc.

Example 5: Suppose we want to find an equation which is satisfied by $\tan\theta$ when $4\theta = \pi/2$. Following the same approach as in example 5, and since we want an expression involving 4θ, we have

$$\begin{aligned}
\cos 4\theta + i\sin 4\theta &= (c+is)^4\,, \\
&= c^4 + 4c^3(is) + 6c^2(is)^2 + 4c(is)^3 + (is)^4, \\
&= c^4 - 6c^2s^2 + s^4 + i(4c^3s - 4cs^3)\,.
\end{aligned}$$

Hence

$$\begin{aligned}
\tan 4\theta &= \frac{\sin 4\theta}{\cos 4\theta}, \\
&= \frac{4c^3s - 4cs^3}{c^4 - 6c^2s^2 + s^4}, \\
&= \frac{4t - 4t^3}{1 - 6t^2 + t^4}\,.
\end{aligned}$$

When $4\theta = \pi/2$, $\tan 4\theta = \infty$, and this can only be so if the denominator of the above equation is zero. Therefore the equation which is satisfied by $\tan\theta$ in this case is $1 - 6t^2 + t^4 = 0$. As an exercise, try finding similar equations for $\tan\theta$ when $3\theta = \pi/2$, $5\theta = \pi/2$, etc.

Example 6: We now know how to use the expressions $z^n \pm 1/z^n$ to find particular trig identities. For example, we know how to express $z^3 + 1/z^3$ in terms of powers of $\cos\theta$ and/or $\sin\theta$. However, suppose we want to express $z^3 + 1/z^3$ in terms of some other form of $z + 1/z$. There are then two ways we could do this: i) knowing the form in which we want to express $z^3 + 1/z^3$, or ii) not knowing the form $z^3 + 1/z^3$ can be expressed in, and therefore having to determine the form.

In the case of i) let us suppose we want to express $z^3 + 1/z^3$ exactly in terms of $z + 1/z$. Then we can start our analysis with $z + 1/z$ (i.e. the form we want to get), and proceed from there, viz:

$$\left(z + \frac{1}{z}\right)^3 = z^3 + 3z^2\left(\frac{1}{z}\right) + 3z\left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3,$$

$$\left(z+\frac{1}{z}\right)^3 \quad = \quad \left(z^3+\frac{1}{z^3}\right)+3\left(z+\frac{1}{z}\right),$$

from which

$$z^3+\frac{1}{z^3}=\left(z+\frac{1}{z}\right)^3-3\left(z+\frac{1}{z}\right).$$

Effectively we have used the "answer", i.e. the form we wanted to recast z^3+1/z^3 into, in order to derive z^3+1/z^3.

However, there are times when we don't know whether a given expression can be reduced into a specific form. There is no "answer" that we can use to start our analysis. In this case we end up having to derive the final form directly from z^3+1/z^3. As such we proceed as follows: letting $\cos\theta+i\sin\theta\equiv c+is$ we have

$$\begin{aligned}
z^3+\frac{1}{z^3} \quad = \quad 2\cos 3\theta \quad &= \quad 2Re(\cos 3\theta+i\sin 3\theta)\,,\\
&= \quad 2Re(c+is)^3\,,\\
&= \quad 2Re(c^3+3c^2(is)+3c(is)^2+(is)^3)\,,\\
&= \quad 2\cos^3\theta-6\cos\theta\,.\sin^2\theta\,,\\
&= \quad 2\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)^3-6\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)\left(\left(\frac{1}{2i}\right)^2\left(z-\frac{1}{z}\right)^2\right),\\
&= \quad \frac{1}{4}\left(z+\frac{1}{z}\right)^3+\frac{3}{4}\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)^2,\\
&= \quad \frac{1}{4}\left(z+\frac{1}{z}\right)^3+\frac{3}{4}\left(z+\frac{1}{z}\right)\left(z^2+\frac{1}{z^2}-2\right),
\end{aligned}$$

Multiplying out the second term on the RHS and collecting terms appropriately we get

$$z^3+\frac{1}{z^3}=\frac{1}{4}\left(z+\frac{1}{z}\right)^3+\frac{3}{4}\left(z^3+\frac{1}{z^3}\right)+\frac{3}{4}\left(z+\frac{1}{z}\right)-\frac{3}{2}\left(z+\frac{1}{z}\right).$$

Moving the second term on the RHS to the LHS, and simplifying the last two terms gives

$$\frac{1}{4}\left(z^3+\frac{1}{z^3}\right)=\frac{1}{4}\left(z+\frac{1}{z}\right)^3-\frac{3}{4}\left(z+\frac{1}{z}\right),$$

Hence

$$z^3 + \frac{1}{z^3} = \left(z + \frac{1}{z}\right)^3 - 3\left(z + \frac{1}{z}\right).$$

Left as an exercise: Express $(z^4 - 1/z^4)/(z - 1/z)$ in terms of $(z - 1/z)^3$.

1.16.4 An alternative approach to deriving the identity for $\tan(n\theta)$

This example is taken from "Where there is pattern, there is significance", Lloyd Olson, *The college mathematics journal*, Vol 20, issue 4 (Sept 1989), p321.

Consider using the above properties of complex numbers to derive the identities for $\tan 2\theta$, $\tan 3\theta$, $\tan 4\theta$, etc. Letting $t \equiv \tan A$ we then have

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{Im(\cos 2\theta + i\sin 2\theta)}{Re(\cos 2\theta + i\sin 2\theta)} = \frac{Im(c+is)^2}{Re(c+is)^2} = \frac{2t}{1-t^2}$$

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{Im(\cos 3\theta + i\sin 3\theta)}{Re(\cos 3\theta + i\sin 3\theta)} = \frac{Im(c+is)^3}{Re(c+is)^3} = \frac{3t-t^3}{1-3t^2}$$

$$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{Im(\cos 4\theta + i\sin 4\theta)}{Re(\cos 4\theta + i\sin 4\theta)} = \frac{Im(c+is)^4}{Re(c+is)^4} = \frac{4t-4t^3}{1-4t^2+t^4}$$

If we look carefully at the way in which the coefficients are distributed across the denominators and the numerator we can see the pattern illustrated in the diagram below.

Tangent identity	*Pascal's Triangle*
	1
	1 1
$\tan 2\theta = \dfrac{2t}{1-1t^2}$	1 2 1
$\tan 3\theta = \dfrac{3t-1t^3}{1-3t^2}$	1 3 3 1
$\tan 4\theta = \dfrac{4t-4t^3}{1-4t^2+1t^4}$	1 4 6 4 1

If we now consider the expansion of $(1+it)^n$, for $n \geq 2$ we have

$$(1+it)^2 = 1 + 2it - t^2$$

$$(1+it)^3 = 1 + 3it - 3t^2 - 3it^3$$

$$(1+it)^4 = 1 + 4it - 6t^2 - 4it^3 + t^4$$

So it seems that the coefficients of $\tan nA$ can be derived from the expansion of $(1+it)^n$, where $n \geq 2$. All we have to do is to take the *Re* and *Im* parts of each of these expressions and divide appropriately. For example, $\tan 3A = Im(1+it)^3/Re(1+it)^3$. So in general we have that

$$(1+it)^n = (1+i.\tan\theta)^n = \left(1 + i.\frac{\sin\theta}{\cos\theta}\right)^n = \left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n = \frac{\cos n\theta + i\sin n\theta}{\cos^n\theta},$$

by DeMoivre's theorem. From this we have

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta} = \frac{Im(1+it)^n}{Re(1+it)^n} = \frac{\binom{n}{1}t - \binom{n}{3}t^3 + \cdots}{1 - \binom{n}{2}t^2 + \binom{n}{4}t^4 + \cdots},$$

where $t \equiv \tan\theta$.

1.16.5 A Proof without words

The following is taken directly from " Proof without Words: Complex Numbers with Modulus One", Jean Huang (Senior), *Mathematics Magazine*, Vol. 79, No. 4 (Oct., 2006), p. 280.

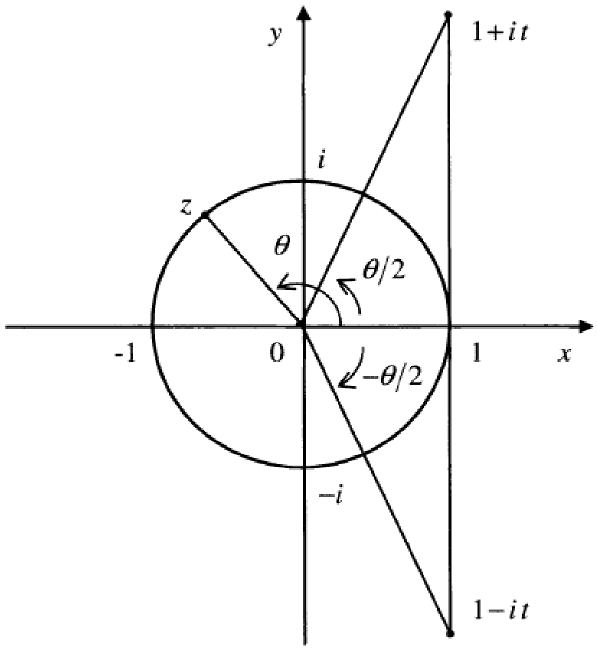
Any complex number z with $|z| = 1$, except $z = -1$, can be expressed as $(1+it)/(1-it)$ for some real number t.

From the diagram below, noting that $t = \tan(\theta/2)$, we see that

$$\arg\left(\frac{1+it}{1-it}\right) = \arg(1+it) - \arg(1-it) = \frac{\theta}{2} - \left(-\frac{\theta}{2}\right) = \theta = \arg(z),$$

and

$$\left|\frac{1+it}{1-it}\right| = \frac{|1+it|}{|1-it|} = 1 = |z|.$$



1.16.6 Finding exact values to trig and inverse trig equations

DeMoivre's theorem can be used very effectively to find exact values to trig trig equation. Going straight into an example will illustrate how we can do this.

<u>*Example 1:*</u> By finding the sum of the roots of $z^3 - 1 = 0$ we can derive an exact value for whatever trig equation arise from this. Hence, for $z^3 = 1$ we have $z_k = \cos(2k\pi/3) + i\sin(2k\pi/3)$ for $k = 0, 1, 2$. Hence

$$z_0 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right),$$

$$z_1 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right),$$

$$z_2 = \cos\left(\frac{6\pi}{3}\right) + i\sin\left(\frac{6\pi}{3}\right) = \cos(2\pi) + i\sin(2\pi) = 1\,.$$

Summing these gives

$$z_0 + z_1 + z_2 = 1 + 2\cos\left(\frac{2\pi}{3}\right).$$

By the analysis of classical algebra relating to $ax^3 + bx^2 + cx + d = 0$ we know that the sum of roots equals $-b/(2a)$. This also applies to polynomial equations involving complex variables. So for $z^3 - 1 =$ we have $-b/(2a) = 0$. Hence

$$1 + 2\cos\left(\frac{2\pi}{3}\right) = 0\,,$$

implying

$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}.$$

So it is possible to use the analysis of complex roots to obtain exact values of trig functions in the real domain.

Example 2: Repeating the analysis above for $z^4 + 1 = 0$ we have $z^4 = -1$ from which the roots are

$$z_k = \cos\frac{\pi + 2k\pi}{4} + i\sin\frac{\pi + 2k\pi}{4},$$

for $k = 0, 1, 2, 3$. Hence

$$z_0 = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right),$$

$$z_1 = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{3}\right),$$

$$z_2 = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right) - i\sin\left(\frac{3\pi}{4}\right),$$

$$z_3 = \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right).$$

Summing these, and knowing that the sum of roots equals zero, we obtain

$$z_0 + z_1 + z_2 + z_3 = \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right) = 0.$$

Exercise: Obtain similar results as above for $z^5 - 1 = 0$, $z^6 - 1 = 0$, etc., and $z^3 + 1 = 0$, $z^4 + 1 = 0$, etc.

Example 3: Consider wanting to derive the exact value of the inverse tan equation relating to $z = 4 + 2i$. How do we do this? Well, our aim here will be to factorise z in such a way that the argument of one of the factors is a standard, known, argument. For z we have

$$4 + 2i = (3 - i)(1 + i).$$

Finding the arguments of both LHS and RHS, and remembering that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ as well as the principal argument interval of $-\pi < \theta \leq \pi$, we have

$$\tan^{-1}\left(\frac{2}{4}\right) = \tan^{-1}\left(-\frac{1}{3}\right) + \tan^{-1}(1).$$

Since $\tan^{-1} 1 = \pi/4$ is a standard result, we have, after rearrangement and simplification,

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}.$$

Note that other complex numbers also gives this same equation. For example, $z = 5 + 5i$ can be factored as $(2 + i)(3 + i)$. Taking the arguments of these numbers we have

$$\begin{aligned} \tan^{-1}\left(\frac{5}{5}\right) &= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right), \\ \text{i.e.} \qquad \frac{\pi}{4} &= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right). \end{aligned}$$

So by considering the arguments of a complex number and its factors we can determine a relationship between the arguments, and find inverse tan equations which satisfy an exact value.

Example 4: Let $z = 3 + 5i$. Then the factors of z are $(4 + i)(1 + i)$. Hence, taking arguments gives

$$\begin{aligned} \tan^{-1}\left(\frac{5}{3}\right) &= \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}(1), \\ \text{i.e.} \qquad \frac{\pi}{4} &= \tan^{-1}\left(\frac{5}{3}\right) - \tan^{-1}\left(\frac{1}{4}\right). \end{aligned}$$

Exercise: For the following complex numbers find two factors, and use these to show the given inverse trig equations:

a) $z = 2 + 4i$; $\tan^{-1} 2 - \tan^{-1}(1/3) = \pi/4$,

b) $z = 4 + 6i$; $\tan^{-1}(3/2) - \tan^{-1}(1/5) = \pi/4$.

Example 5: Notice that $\cos\pi + i\sin\pi = -1$, $\cos 2\pi + i\sin 2 = 1$, $\cos 3\pi + i\sin 3\pi = -1$, etc., so that in general we have $\cos k\pi + i\sin k\pi = (-1)^k$. We can express the LHS of this last equation in many ways, for example $(\cos k\pi/2 + i\sin k\pi/2)^2$, $(\cos k\pi/3 + i\sin k\pi/3)^3$, etc.

Letting $\cos\theta + i\sin\theta \equiv c + is$, let us analyse $(c + is)^3$ where $\theta = k\pi/3$. Expanding and collecting *Re* and *Im* terms we have

$$(c + is)^3 = c^3 - 3cs^2 + i(3c^2s - s^3) = (-1)^k.$$

Comparing *Im* terms left and right we have

$$3c^2s - s^3 = s(3c^2 - s^2) = 0\,.$$

Therefore $\sin(k\pi/3) = 0$ or $3\cos^2(k\pi/3) - \sin^2(k\pi/3) = 0$. Since $\sin(k\pi/3) = 0$ is not generally valid we have

$$3\cos^2\left(\frac{k\pi}{3}\right) - \sin^2\left(\frac{k\pi}{3}\right) = 4\cos^2\left(\frac{k\pi}{3}\right) - 1 = 0\,.$$

Letting $k = 1, 2, 3$ we have

- For $k = 1$, $4\cos^2(\pi/3) - 1 = 0$, implying $\cos(\pi/3) = \pm 1/2$. Since $\pi/3$ is in the first quadrant we end up with

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}\,.$$

- For $k = 2$, $4\cos^2(2\pi/3) - 1 = 0$, implying $\cos(2\pi/3) = \pm 1/2$. Since $2\pi/3$ is in the second quadrant we end up with

$$\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}\,.$$

- For $k = 3$, $4\cos^2(3\pi/3) - 1 = 0$, implying $\cos(3\pi/3) = \pm 1/2$. Since $3\pi/3$ is not in the first quadrant we end up with

$$\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}\,.$$

Hence we have (amongst other possibilities) the following exact-value equations:

$$\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{2\pi}{3}\right) = 0\,, \qquad \cos\left(\frac{2\pi}{3}\right) - \cos\left(\frac{\pi}{3}\right) = -1\,, \qquad \cos\left(\frac{2\pi}{3}\right) + \cos(\pi) = -1\,.$$

We can repeat the above analysis for $(c + is)^n$ where $\theta = k\pi/n$, for any value $n \in \mathbb{N}$. Therefore, for $(c + is)^5 = (-1)^k$ we have

$$(c + is)^5 = c^5 - 10c^3s^2 + 5cs^4 + i(5c^4s - 10c^2s^3 + s^5) = (-1)^k\,.$$

Comparing *Im* terms left and right we have

$$5c^4s - 10c^2s^3 + s^5 = 0\,.$$

Since $\sin(k\pi/5) \neq 0$ in general we can divide the above equation by this term, giving

$$5c^4 - 10c^2s^2 + s^4 = 0\,.$$

Substituting $s^2 = 1 - c^2$, and simplifying, we obtain

$$16c^4 - 12c^2 + 1 = 0\,.$$

This equation is a quadratic in c^4 hence solving by the quadratoc formula we end up with

$$\cos\left(\frac{k\pi}{5}\right) = \pm\sqrt{\frac{3 \pm \sqrt{5}}{8}} = \pm\frac{1 \pm \sqrt{5}}{4}\,.$$

There are four values here with alternating $\pm$ signs. Knowing that as θ increases from $0 \to \pi$, $\cos\theta$ decreases from $1 \to -1$, the values above are in the following order for $k = 1, 2, 3, 4$:

- For $k = 1$,
$$\cos\left(\frac{\pi}{5}\right) = \frac{1 + \sqrt{5}}{4}\,.$$
- For $k = 2$,
$$\cos\left(\frac{2\pi}{5}\right) = \frac{-1 + \sqrt{5}}{4}\,.$$
- For $k = 3$,
$$\cos\left(\frac{3\pi}{5}\right) = \frac{1 - \sqrt{5}}{4}\,.$$
- For $k = 4$,
$$\cos\left(\frac{4\pi}{5}\right) = \frac{-1 - \sqrt{5}}{4}\,.$$

Hence we have (amongst other possibilities) the following exact-value equations:

$$\cos\left(\frac{\pi}{5}\right) - \cos\left(\frac{2\pi}{5}\right) = \frac{1}{2}\,, \qquad \cos\left(\frac{3\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = \frac{\sqrt{5}}{2}\,,$$

$$\cos\left(\frac{4\pi}{5}\right) - \cos\left(\frac{\pi}{5}\right) = -\frac{1}{2} - \frac{\sqrt{5}}{2}\,, \qquad \cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{3\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = 0\,.$$

Example 6: Using the method of section 1.16.2 we can show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. If we let $\theta = \pi/8$ we obtain

$$\cos 4\left(\frac{\pi}{8}\right) = 8\cos^4\left(\frac{\pi}{8}\right) - 8\cos^2\left(\frac{\pi}{8}\right) + 1\,,$$

implying

$$8\cos^4\left(\frac{\pi}{8}\right) - 8\cos^2\left(\frac{\pi}{8}\right) + 1 = 0\,.$$

Solving this as a quadratic in $\cos^2(\pi/8)$ we have $\cos^2(\pi/8) = \left(2 \pm \sqrt{2}\right)/4$ giving us $\cos(\pi/8) = \pm\left(\sqrt{2 \pm \sqrt{2}}\right)/2$. Since $\pi/8$ in the first quadrant, the only possible answers are $\cos(\pi/8) = \left(\sqrt{2 \pm \sqrt{2}}\right)/2$, and testing both these answers shows us that the only valid answer is

$$\cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2 + \sqrt{2}}}{2}.$$

Similar substitution with appropriate values for θ can be made in other trig equations involving multiple angles.

1.17 On the connection between general roots of a complex number and roots of unity – To come

1.18 Exponential form of a complex number

1.18.1 Euler's formula

So far we have seen two ways in which a complex number can be represented, namely the Cartesian form $z = x + iy$ and the polar form $z = r(\cos\theta + i\sin\theta)$. We will now see a third way of expressing a complex number.

At this stage in one's mathematical development the most usual way of developing this third way is to consider the Taylor series for $\cos x$ and $\sin x$, and compare these with the Taylor series for e^x. Hence

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad , \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

and

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x}{6!} + \frac{x}{7!} + \cdots$$

These three series converges for all values of θ in $-\infty < \theta < \infty$. Now, if we look carefully we might feel that we can combine the series for $\cos\theta$ and $\sin\theta$ in some way so as to obtain the series for e^θ. This can indeed be done if we remember our powers of i, namely that $i = i$, $i^2 = -1$, $i^3 = -i$, etc. Then, if we let $x = i\theta$ in the expeonential series we have

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots, \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \cdots, \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right). \end{aligned}$$

Hence we see that

$$e^{i\theta} = \cos\theta + i\sin\theta\ . \tag{51}$$

This formula is known as Euler's formula/identity for complex numbers (don't confuse this with the other Euler formula which links the number of vertices and edges of a polygon). What (51) shows us is that, by moving from the real domain to the complex domain, we are able to establish a connection between the exponential function and the two main trig functions.

However, we now have a fundamental problem: although we have been able to use i to arithmetically combine the cos and sin series into the exponential series is it just a coincidence, a quirk of arithmetic? Another way of saying this is, How do we know if (51) converges? Remember that not all arithmetic on real numbers transfers automatically to arithmetic on complex numbers, so just because the cos, sin and exponential series converge for real values it doesn't mean that this will automatically be so for complex values. For example,

$$\text{in } \mathbb{R} \text{ the sum } 1+2+3+4+\cdots \to \infty \text{ (diverges)}$$

whereas

$$\text{in } \mathbb{C} \text{ the sum } 1+2+3+4+\cdots = -\tfrac{1}{12} \text{ (converges)}$$

This last one, seeming very strange (if not impossible), would require us to go into some high level complex analysis about the converges characteristics of complex valued series, and something called the Riemann Zeta function. Since this lies way beyond the scope of these notes we will go through two alternative ways of illustrating that (51) is indeed true for all θ.

The first approach is taken from "Complex numbers in advanced algebra", by H. E. Webb, *American mathematical monthly*, Vol 27, No 11, (Nov 1920), pp411-413: Let us assume that $e^{z\theta} = \cos\theta + i\sin\theta$ for some complex number z. Then, since $e^{z\theta}.e^{-z\theta} = 1$ and $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = 1$ we can write $e^{-z\theta} = \cos\theta - i\sin\theta$. Adding and subtracting the equations in $e^{z\theta}$ and $e^{-z\theta}$ we have

$$\cos\theta = \frac{e^{z\theta}+e^{-z\theta}}{2} \qquad , \qquad i\sin x = \frac{e^{z\theta}-e^{-z\theta}}{2}.$$

Now, if we expand $e^{z\theta}$ and $e^{-z\theta}$ according to their Taylor series, and then halve these, we obtain

$$\cos\theta = 1 + \frac{z^2\theta^2}{2!} + \frac{z^4\theta^4}{4!} + \frac{z^6\theta^6}{6!} + \cdots$$

and

$$i\sin\theta = z\theta + \frac{z^3\theta^3}{3!} + \frac{z^5\theta^5}{5!} + \frac{z^7\theta^7}{7!} + \cdots \qquad (*)$$

Dividing both sides of (*) by θ we obtain

$$i\frac{\sin\theta}{\theta} = z + \frac{z^3\theta^2}{3!} + \frac{z^5\theta^4}{5!} + \frac{z^7\theta^6}{7!} + \cdots$$

Taking the limit as $\theta \to 0$ we have

$$i \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \quad = \quad \lim_{\theta \to 0} \left(z + \frac{z^3\theta^2}{3!} + \frac{z^5\theta^4}{5!} + \frac{z^7\theta^6}{7!} + \cdots \right), \qquad (**)$$

$$\Rightarrow \qquad i \quad = \quad z\,.$$

So even if the sum of the terms after the first term in (**) does diverge for any general θ, this sum will approach 0 as θ approaches 0. Hence we can say that $e^{i\theta} = \cos\theta + i\sin\theta$.

However, by dividing (*) by θ and taking limits we have effectively side-stepped the question of convergence of $e^{i\theta}$. This feels like a bit of a cheat, and the problem of whether or not the complex-valued expansion of $e^{i\theta}$ converges still remains to be addressed (in a future complex analysis course).

One way to overcome this is to develop a more formal and abstract approach. As such let us assume that $e^{i\theta}$ can be formed as a combination of two functions $f(\theta)$ and $g(\theta)$ as follows:

$$e^{i\theta} = f(\theta) + i.g(\theta)\,. \qquad [*]$$

Now, when developing a new mathematical object, such as a complex number, or a function of a complex variable, what we want to do is to keep as many of the arithmetic and calculus rules that work for real numbers and functions. This will not always be possible, so what we do is assume that a certain rule of real numbers and functions can be transferred to complex numbers and functions, apply this rule and see if we are able to obtain something which "works"

So, our question is What form do $f(\theta)$ and $g(\theta)$ take? Can we even find suitable functions $f(\theta)$ and $g(\theta)$? Well, in order to be consistent with current arithmetic and the laws of indices we want $e^0 = 1$. Therefore,

$$1 = f(0) + i.g(0)\,.$$

Comparing *Re* and *Im* parts implies $f(0) = 1$ and $g(0) = 0$. Also, if $e^{i\theta}$ is to have the usual property of differentiation the we need

$$ie^{i\theta} = f'(\theta) + i.g'(\theta)\,.$$

(in other words, we are assuming that we can differentiatie a function having i just as we can differentiate a function having a real number). When $\theta = 0$ we have

$$i = f'(0) + i.g'(0)\,.$$

This is true on condition that $f'(0) = 0$ and $g'(0) = 1$.

Differentiation again gives us

$$-e^{i\theta} = f''(\theta) + i.g''(\theta)\,. \qquad [**]$$

When $\theta = 0$ we have

$$-1 = f''(0) + i.g''(0)\,,$$

again being true on condition that $f''(0) = -1$ and $g'(0) = 0$. Adding [*] and [**] we have

$$f''(\theta) + f(\theta) + i.\big(g''(\theta) + g(\theta)\big) = 0\,,$$

implying

$$f''(\theta) + f(\theta) = 0, \text{with } f(0) = 1 \text{ and } f'(0) = 0\,,$$

and $$g''(\theta) + g(\theta) = 0, \text{with g}(0) = 0 \text{ and } f'(0) = 1\,.$$

These two equations, along with their associated conditions, are standard ordinary differential equations whose solution are also standard: $f(\theta) = \cos\theta$ and $g(\theta) = \sin\theta$. Hence we are correct in making the arithmetic and differentiation assumptions above. Therefore if $e^{i\theta}$ is to be a complex number with the arithmetic and differentiation assumptions above then we have $e^{i\theta} = \cos\theta + i\sin\theta$.

<u>*Example 1:*</u> If we let $\theta = \pi$ in Euler's identity we obtain $e^{i\pi} = \cos\pi + i\sin\pi$ which simplifes to

$$e^{i\pi} = -1 \text{ or } e^{i\pi} - 1 = 0.$$

This equation links the fundamental numbers 0, 1, e, and π into one elegant expression.

<u>*Example 2:*</u> To express $z = -10$ in exponential form we find r and θ in the same way as before. Hence we have that $r = |z| = \sqrt{(-10)^2 + 0^2} = 10$, and $\theta = \arg(z) = \tan^{-1}(0/-10) + \pi = \pi$. Hence $z = 10e^{i\pi}$.

<u>*Example 3:*</u> Similar to example 1 we can express $z = -2\pi i$ in exponential form to be $r = |z| = 2\pi$ and $\theta = \arg(z) = \tan^{-1}(-2\pi/0) = -\pi/2$. Hence $z = 2\pi.e^{-i\pi/2}$.

<u>*Example 4:*</u> To express $(1+i)^{20}$ in exponential form we let $z = 1 + i$. Then we have $r = |z| = \sqrt{2}$ and $\theta = \arg(z) = \tan^{-1}(1) = \pi/4$. Hence $z = \sqrt{2}.e^{i\pi/4}$. Therefore $z^{20} = \left(\sqrt{2}.e^{i\pi/4}\right)^{20} = 1024.e^{20i\pi/4} = 1020.e^{i\pi}$.

Example 5: For $z = -2\sqrt{3} - 2i$ in exponential form we have $r = |z| = \sqrt{12+4} = 4$ and $\theta = \arg(z) = \tan^{-1}\left(-2/\left(-2\sqrt{3}\right)\right) - \pi = -5\pi/6$. Hence $z = 4e^{-5i\pi/6}$.

Example 6: To express $z = (1+i)/\left(\sqrt{3} - i\right)$ in exponential form we have

$$r = |z| = \left|\frac{(1+i)}{\left(\sqrt{3}-i\right)}\right| = \frac{|1+i|}{\left|\sqrt{3}-i\right|} = \frac{\sqrt{2}}{2},$$

and

$$\theta = \arg(z) = \arg\left(\frac{1+i}{\sqrt{3}-i}\right) = \arg(1+i) - \arg\left(\sqrt{3}-i\right) = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}.$$

Hence $z = \frac{\sqrt{2}}{2}.e^{i\pi/12}$.

Example 7: To express the complex number $z = e^{-i\pi/3}$ in Cartesian form we simply rewrite it in polar form and evaluate the trig functions. Hence $z = \cos(-i\pi/3) + i\sin(-i\pi/3) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$.

Example 8: Similar to example 6 we can express the complex number $z = 5e^{i\pi}$ in Cartesian form by again rewriting it in polar form and evaluating the trig finctions. Hence: $z = 5(\cos(\pi) + i\sin(\pi)) = -5$.

Exercises: Show that i) $2 + i = \sqrt{5}.e^{i\left(\tan^{-1}(1/2)\right)}$, and ii) $-3 - 4i = 5e^{i\left(\tan^{-1}(4/3) - \pi\right)}$.

1.18.2 Proof of DeMoivre's theorem for all real powers

Euler's identity was derived independently from any mathematics used to derive/prove DeMoivre's theorem. As such we can prove DeMoivre's theorem from Euler's identity. Let us therefore start with Euler's identity:

$$e^{i\theta} = \cos\theta + i\sin\theta\,.$$

Hence, for any $n \in \mathbb{R}$ we have

$$\left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n\,. \qquad (*)$$

Now, although we haven't proved this (since it is beyond the scope of these notes) it is the case that $\left(e^{i\theta}\right)^n$ is defined for all $n \in \mathbb{R}$, and that $\left(e^{i\theta}\right)^n = e^{in\theta}$ (if and when I get to the proofs of these they will be in the notes "Complex numbers III").

Since $n\theta$ can be considered as an angle in-and-of-itself, we can write $e^{in\theta} = e^{i(n\theta)}$, and we can apply Euler's formula again to obtain

$$e^{i(n\theta)} = \cos(n\theta) + i\sin(n\theta)\,. \tag{**}$$

Hence, by (*) and (**) we have

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)\,. \tag{52}$$

Thus we can now apply DeMoivre's theorem for any $n \in \mathbb{R}$. But, it is important to understand that, although it may look it, (52) does not represent the process of directly converting a power into a multiplication, as it does when n is an integer or a rational number. Equation (52) is, in fact, the consequence of Euler's identity and the properties of complex exponential functions. This then allows us to connect the expression $(\cos\theta + i\sin\theta)^n$ to $\cos(n\theta) + i\sin(n\theta)$ in a mathematically valid manner. In brief, the valid sequence of steps which allow (52) to be true are

$$e^{i\theta} = \cos\theta + i\sin\theta\,,$$

therefore,

$$\left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n.$$

But

$$\left(e^{i\theta}\right)^n = e^{in\theta} = e^{i(n\theta)}.$$

So

$$e^{i(n\theta)} = \cos(n\theta) + i\sin(n\theta)\,.$$

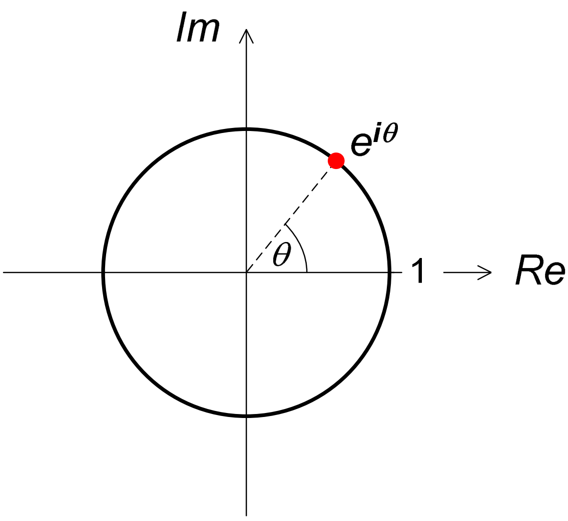
1.18.3 The geometric intepretation of $e^{i\theta}$

Since $e^{i\theta} = \cos\theta + i\sin\theta$ we have $\left|e^{i\theta}\right| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$. Also, since $x = \cos\theta$ and $y = \sin\theta$ we have $\sqrt{x^2 + y^2} = 1$ implying $x^2 + y^2 = 1$. All this means that $e^{i\theta}$ represents a point on a unit circle, centre $(0, 0)$, at an angle θ to the horizontal, as illustrated in diagram (a) below.

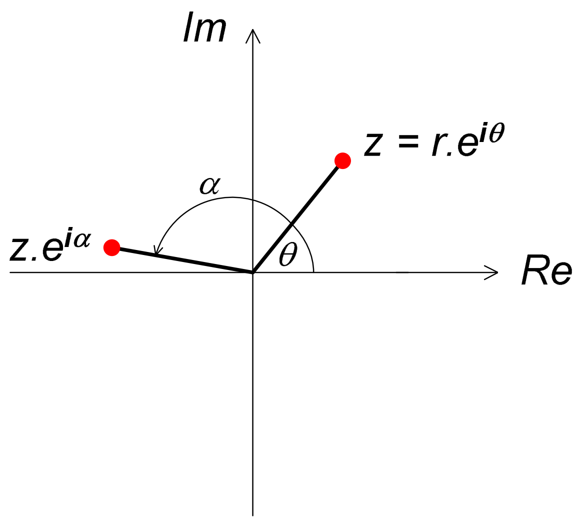
Consider now $z.e^{i\alpha}$, where $\alpha \in \mathbb{R}$. What is the geometric effect on z of multiplying by $e^{i\alpha}$? Well,

$$z.e^{i\alpha} = re^{i\theta}e^{i\alpha} = re^{i(\theta+\alpha)}\,.$$

Hence $e^{i\alpha}$ has the effect of rotating z by an angle α. This is illustrated in diagram (b) below.



(a): $e^{i\theta}$ as apoint on the unit circle



(b): The effect on z of multiplying by $e^{i\alpha}$

1.18.4 The relationship between cos and sin and $e^{i\theta}$

Probably one of the most useful properties of complex numbers in exponential form involves the connection between $e^{i\theta}$ and the cos and sin functions. Therefore, if $z = e^{i\theta}$ and $z^* = e^{-i\theta}$ then

$$z + z^* = \left(e^{i\theta} + e^{-i\theta}\right) = 2\cos\theta\,,$$

and
$$z - z^* = \left(e^{i\theta} - e^{-i\theta}\right) = 2i\sin\theta\,.$$

Here we have two equations which relate the trig functions to the exponential form of a complex number, namely

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right), \qquad \sin\theta = \frac{1}{2i}\left(e^{i\theta} + e^{-i\theta}\right). \tag{53}$$

These relations allow us to more easily manipulate complex numbers and trig equations. More generally it can be shown that

$$\cos k\theta = \frac{1}{2}\left(e^{ik\theta} + e^{-ik\theta}\right), \qquad \sin k\theta = \frac{1}{2i}\left(e^{ik\theta} + e^{-ik\theta}\right). \tag{54}$$

Now, just as we used the relationship between complex numbers and trig functions in section 1.16, to derive trig identities, so we can use (53) and/or (54) above to derive trig identities. So, to derive the identity for $\sin^3\theta$ we proceed as follows:

$$\begin{aligned}
\sin^3\theta &= \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^3, \\
&= -\frac{1}{8i}\left\{\left(e^{i\theta}\right)^3 - 3\left(e^{i\theta}\right)^2\left(e^{-i\theta}\right) + 3\left(e^{i\theta}\right)\left(e^{-i\theta}\right)^2 - \left(e^{-i\theta}\right)^3\right\}, \\
&= -\frac{1}{8i}\left(e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}\right).
\end{aligned}$$

Collecting terms in like powers we obtain

$$\sin^3\theta = -\frac{1}{4}\left(\frac{e^{3i\theta}-e^{-3i\theta}}{2i}\right) + \frac{3}{4}\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right),$$

which gives

$$\sin^3\theta = +\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta\,.$$

We can also use (53) and/or (54) above to derive standard trig identities such as

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

as follows:

$$\begin{aligned}
2\sin A\cos B &= 2\left(\frac{e^{iA}-e^{-iA}}{2i}\right)\left(\frac{e^{iB}+e^{-iB}}{2}\right), \\
&= \frac{1}{2i}\left(e^{iA}e^{iB} + e^{iA}e^{-iB} - e^{-iA}e^{iB} - e^{-iA}e^{-iB}\right), \\
&= \frac{1}{2i}\left(e^{i(A+B)} + e^{i(A-B)} - e^{-i(A-B)} - e^{-i(A+B)}\right), \\
&= \frac{1}{2i}\left[\left(e^{i(A+B)} - e^{-i(A+B)}\right) + \left(e^{i(A-B)} - e^{-i(A-B)}\right)\right], \\
&= \sin(A+B) + \sin(A-B)\,.
\end{aligned}$$

1.18.5 More examples

<u>*Example 1:*</u> Evaluating $z = (3e^{i\pi/6})(2e^{-5i\pi/4})$ we have $z = 6e^{i\pi/6-5\pi/4} = 6e^{-13i\pi/12}$. Converting this into principal argument form we obtain $z = 6e^{-i\pi/12}$.

<u>*Example 2:*</u> Evaluating $z = 6e^{5i\pi/3}/(4e^{2i\pi/3})^2$ we have $z = \frac{6}{16}e^{5i\pi/3}/e^{4i\pi/3} = \frac{3}{8}e^{i\pi/3}$.

<u>*Example 3:*</u> Evaluating $z = \left((\sqrt{3}-i)/(\sqrt{3}+i)\right)^4 \times ((1+i)/(1-i))^5$ we have first convert to exponential form: $z = (2e^{-i\pi/6}/2e^{i\pi/6})^4(\sqrt{2}e^{i\pi/4}/\sqrt{2}e^{-i\pi/4})^5$. We now simplify and use the rules of exponents: $z = (e^{-2i\pi/6})^4(e^{2i\pi/6})^5 = e^{7i\pi/6} = e^{i\pi/6}$.

<u>*Example 4:*</u> If $z = 2\sqrt{2} + 2\sqrt{2}\,i$ and $w = -1.5\sqrt{3} + 1.5i$ then we can find z^2w^4 in exponential form as follows: $r_1 = |z| = 4$ and $\theta = \arg(z) = \tan^{-1}[(2\sqrt{2})/(2\sqrt{2})] = \pi/4$, hence $z = 4e^{i\pi/4}$. Also, $r_2 = |w| = 3$, and $\theta = \arg(w) = \tan^{-1}[1.5/(-1.5\sqrt{3})] + \pi = 5\pi/6$, hence $w = 3e^{5i\pi/6}$.

Therefore

$$\begin{aligned} z^2w^4 &= (4e^{i\pi/4})^2(3e^{5i\pi/6})^4\,, \\ &= (16)(81)(e^{2i\pi/4})(e^{20i\pi/6})\,, \\ &= 1296e^{23i\pi/6}\,, \\ &= 1296e^{5i\pi/6}\,. \end{aligned}$$

<u>*Example 5:*</u> If $z = \frac{5}{6} + \frac{5}{6}\sqrt{3}\,i$ and $w = -4\sqrt{3} - 4i$ then we can find $(z^3w^4)^*$ in exponential form as follows: $r_1 = |z| = 5/3$ and $\theta = \arg(z) = \tan^{-1}\sqrt{3} = \pi/3$, hence $z = 5e^{i\pi/3}/3$. Also, $r_2 = |w| = 8$, and $\theta = \arg(w) = \tan^{-1}[-4/(-4\sqrt{3})] - \pi = -5\pi/6$, hence $w = 8e^{-5i\pi/6}$.

Therefore

$$\begin{aligned} (z^3w^4)^* &= \left[\left(\frac{5}{3}e^{i\pi/3}\right)^3 (8e^{-5i\pi/6})^4\right]^*\,, \\ &= \left[\left(\frac{5}{3}\right)^3 8^4 e^{i\pi}.e^{-10i\pi/3}\right]^*\,, \end{aligned}$$

$$\text{So } (z^3w^4)^* = 8\left(\frac{40}{3}\right)^3 e^{7i\pi/3},$$

$$= 8\left(\frac{40}{3}\right)^3 e^{i\pi/3}.$$

Example 6: If $z = -\frac{3}{4}\sqrt{2} + \frac{3}{2}\sqrt{2}\,i$ and $w = 0.3 - 0.3\sqrt{3}\,i$ then we can find z^*/w^3 in exponential form as follows: $r_1 = |z| = 3\sqrt{10}/4$ and $\theta = \arg(z) = \tan^{-1}(-2) + \pi \approx 2.034$ radians, hence $z = 3\sqrt{10}.\, e^{2.034i}/4$. Also, $r_2 = |w| = 0.6$, and $\theta = \arg(w) = \tan^{-1}\left[-1/\sqrt{3}\right] = -\pi/6$, hence $w = 0.6e^{-i\pi/6}$.

Therefore

$$\frac{z^*}{w^3} = \frac{\left(\frac{3\sqrt{10}}{4}e^{2.034i}\right)^*}{\left(\frac{3}{5}e^{-i\pi/6}\right)^3},$$

$$= \frac{\frac{3\sqrt{10}}{4}e^{-2.034i}}{\frac{27}{125}e^{-i\pi/2}},$$

$$= \frac{125\sqrt{10}}{36}\left(e^{-2.034i}\right)\left(e^{i\pi/2}\right),$$

$$= \frac{125\sqrt{10}}{36}.\, e^{-0.463i}.$$

Example 7: Suppose we want to show

$$\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \sin\frac{6\pi}{n} + \cdots + \sin\frac{2(n-1)\pi}{n} = 0.$$

How do we recognise that we can use complex analysis to solve this problem? Well, one sign is that we are dividing each angle by the same value n. This suggests that we are taking the n^{th} root of something. Also notice that each successive numerator goes like $2k\pi$. So it seems as if we are taking the n^{th} root of unity. Hence it might be worth trying to use complex analysis for this problem. So let us solve $z^n = 1$. As usual, $r = |z| = 1$ and $\theta = \arg(z) = 2k\pi$ for $k = 0, \pm 1, \pm 2, \ldots$ So we have

$$z^n = e^{2ki\pi}\,.$$

Therefore

$$z = e^{2ki\pi/n}\,.$$

Hence we have roots

$$z_0 = 1, z_1 = e^{2i\pi/n}\,, z_2 = e^{4i\pi/n}, z_3 = e^{6i\pi/n}, \ldots, z_{n-1} = e^{2(n-1)i\pi/n}\,.$$

Since the sum of the roots of $z^n - 1 = 0$ is zero we have

$$1 + e^{2i\pi/n} + e^{4i\pi/n} + e^{6i\pi/n} + .. + e^{2(n-1)i\pi/n} = 0\,,$$

from which (by comparing *Re* and *Im* parts)

$$\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \sin\frac{6\pi}{n} + \cdots + \sin\frac{2(n-1)\pi}{n} = 0\,.$$

1.18.6 Roots of complex numbers in exponential form

Finding roots via the exponential form of a complex number can also be done. In this case if $z = r.e^{i\theta}$ then

$$z^{1/n} = r^{1/n}.\left(e^{i(\theta+2k\pi)}\right)^{1/n} = r^{1/n}.e^{i(\theta+2k\pi)/n}\,, \qquad (55)$$

where $k = 0, \pm 1, \pm\, 2, \pm 3 \ldots$, and where the principal value is given by $z = \sqrt[n]{r}.e^{i\theta/n}$, with the interval for the principal argument still being $-\pi < \mathrm{Arg}(z) \leq 1$.

<u>*Example 1:*</u> To find the four 4th roots of $z = 1 + i$ in exponential form we have $r = |z| = \sqrt{2}$ and $\theta = \arg(z) = \pi/4$. Hence $z = \sqrt{2}.e^{i\pi/4}$, and $z^{1/4} = \left(\sqrt{2}.e^{i(\pi/4+2k\pi)}\right)^{1/4} = \sqrt[8]{2}.e^{i(\pi/4+2k\pi)/4}$. Hence

$$\begin{aligned} z_0 &= \sqrt[8]{2}.e^{i\pi/16}\,, & z_1 &= \sqrt[8]{2}.e^{9i\pi/16}\,, \\ z_2 &= \sqrt[8]{2}.e^{17i\pi/16}\,, & z_3 &= \sqrt[8]{2}.e^{25i\pi/16}\,. \end{aligned}$$

<u>*Example 2:*</u> To find the six 6th roots of $z = -1 - \sqrt{3}i$ in exponential form we have $r = |z| = 2$ and $\theta = \arg(z) = \tan^{-1}\left(-\sqrt{3}\right)/(-1) - \pi = -2\pi/3$. Hence $z = 2.e^{-2i\pi/3}$, and $z^{1/6} = \left(2.e^{i(-2\pi/4+2k\pi)}\right)^{1/6} = \sqrt[8]{2}.e^{i(-2\pi/3+2k\pi)/6}$.

Therefore

$$z_0 = \sqrt[8]{2}.\, e^{-2i\pi/18}\,, \qquad z_1 = \sqrt[8]{2}.\, e^{4i\pi/18}\,,$$

$$z_2 = \sqrt[8]{2}.\, e^{10i\pi/18}\,, \qquad z_3 = \sqrt[8]{2}.\, e^{16i\pi/18}\,,$$

$$z_4 = \sqrt[8]{2}.\, e^{22i\pi/18} = \sqrt[8]{2}.\, e^{-14i\pi/18}\,, \qquad z_5 = \sqrt[8]{2}.\, e^{28i\pi/18} = \sqrt[8]{2}.\, e^{-8i\pi/18}\,.$$

1.18.7 More complicated examples

1) Let us solve

$$\left(\frac{z+i}{z-i}\right)^n = 1\,.$$

We have seen examples similar to this in section 1.15, but we will now solve this using the exponential form of complex numbers. Hence

$$\frac{z+i}{z-i} = 1^{1/n} = (e^{2k\pi})^{1/n} = e^{2k\pi/n}\,,$$

for $k = 0, 1, 2, \dots, n-1$. Therefore $z + i = (z-i)e^{2k\pi/n}$. Expanding and collecting terms in z we obtain $z\left(e^{2k\pi/n} - 1\right) = i\left(e^{2k\pi/n} + 1\right)$. Hence

$$\begin{aligned} z &= i.\frac{e^{2k\pi/n}+1}{e^{2k\pi/n}-1}, \\ &= i.\frac{e^{k\pi/n}+e^{-k\pi/n}}{e^{k\pi/n}-e^{k\pi/n}}, \\ &= i.\frac{2\cos k\pi/n}{2i\sin k\pi/n}. \end{aligned}$$

Hence $z = \cot k\pi/n$. Notice how much easier it is to perform complex number work in exponential form compared to that of section 1.15.

2) Suppose we want to find the sum of

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta\,.$$

Knowing about complex numbers we can rewrite this as $Re\left(1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta}\right)$. Now notice that this is a geometric sequence with first term equal to 1 and common ratio equal to $e^{i\theta}$.

We can therefore, find the sum of this series by the usual geometric sum formula. Hence

$$Re\left(1+e^{i\theta}+e^{2i\theta}+\cdots+e^{ni\theta}\right)=Re\left(\frac{1-e^{11i\theta}}{1-e^{i\theta}}\right).$$

We now want to factor out a term in $e^{11i\theta/2}$ so as to transform the numerator into its equivalent cos and sin forms. Ditto for the denominator. So we now have

$$\begin{aligned} Re\left(1+e^{i\theta}+e^{2i\theta}+\cdots+e^{ni\theta}\right) &= Re\left(\frac{e^{11i\theta/2}\left(e^{-11i\theta/2}\ -e^{11i\theta/2}\right)}{e^{i\theta/2}\left(e^{-i\theta/2}\ -e^{i\theta/2}\right)}\right), \\ &= Re\left(.\,e^{5i\theta}\frac{\sin 11\theta/2}{\sin\theta/2}\right), \\ &= \cos 5\theta\,.\frac{\sin 11\theta/2}{\sin\theta/2} \end{aligned}$$

<u>Exercise:</u> Find the sum of $\cos\theta+\cos 2\theta+\cdots+\cos n\theta$. This is the same sum as example 2, but without th eleading 1 (ans: $\cos\left((n+1)\theta/2\right)(\sin n\theta/2)/(\sin\theta/2)$).

<u>Exercise:</u> Find the sum of $1+\sin\theta+\sin 2\theta+\cdots+\sin n\theta$, and $\sin\theta+\sin 2\theta+\cdots+\sin n\theta$

3) Continuing the idea of example 2 we can find the sum of $\cos\theta+\cos(\theta+\alpha)+\cos(\theta+2\alpha)\ldots+\cos\cos(\theta+n\alpha)$:

$$Re\left(e^{i\theta}+e^{i(\theta+\alpha)}+e^{i(\theta+2\alpha)}+\cdots+e^{i(\theta+n\alpha)}\right).$$

Factoring out $e^{i\theta}$ we obtain

$$Re\left\{e^{i\theta}\left(1+e^{i\alpha}+e^{2i\alpha}+\cdots+e^{ni\alpha}\right)\right\}.$$

The terms in the bracket form a geometric series with first term 1 and common ratio $e^{i\alpha}$, from which th eresult of the solution can be developed. This is left as an exercise.

4) Consider studying $\left(1+e^{2i\theta}\right)^n$ in two different ways. The first way is to do a standard algebraic trick as follows:

$$\left(1+e^{2i\theta}\right)^n = \left[\left(1+e^{2i\theta}\right)\left(\frac{e^{-i\theta}}{e^{-i\theta}}\right)\right]^n,$$

$$\text{So } \left(1+e^{2i\theta}\right)^n = \left(e^{-i\theta}+e^{i\theta}\right)^n\left(e^{i\theta}\right)^n,$$

$$= 2\cos\theta\,(\cos n\theta + i\sin n\theta)\,.$$

The second way is to expand $\left(1+e^{2i\theta}\right)^n$ using the binomial theorem. Now, for two complex numbers z and w we have $(z+w)^n = \sum \binom{n}{k} z^{n-k}w^k$. Hence

$$\begin{aligned}
\left(1+e^{2i\theta}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\left(e^{2i\theta}\right)^k, \\
&= \sum_{k=0}^{n}\binom{n}{k}(\cos 2\theta + i\sin 2\theta)^k, \\
&= 1+\sum_{k=1}^{n}\binom{n}{k}(\cos 2\theta + i\sin 2\theta)^k, \\
&= 1+\sum_{k=1}^{n}\binom{n}{k}(\cos 2k\theta + i\sin 2k\theta), \\
&= 1+\sum_{k=1}^{n}\binom{n}{k}\cos 2k\theta + i\sum_{k=1}^{n}\binom{n}{k}\sin 2k\theta,
\end{aligned}$$

Comparing *Re* and *Im* parts of both versions of $\left(1+e^{2i\theta}\right)^n$ we have

$$Re:\quad 2\cos\theta\cos n\theta = 1+\sum_{k=1}^{n}\binom{n}{k}\cos 2k\theta\,,$$

$$Im:\quad 2\cos\theta\sin n\theta = \sum_{k=1}^{n}\binom{n}{k}\sin 2k\theta\,,$$

Hence

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta} \quad = \quad \frac{\sum_{k=1}^{n}\binom{n}{k}\sin 2k\theta}{1+\sum_{k=1}^{n}\binom{n}{k}\cos 2k\theta},$$

$$= \quad \frac{\sum_{k=0}^{n}\binom{n}{k}\sin 2k\theta}{\sum_{k=0}^{n}\binom{n}{k}\cos 2k\theta},$$

$$= \quad \frac{n\sin 2\theta + \frac{n(n-1)}{2!}\sin 4\theta + \frac{n(n-1)(n-2)}{3!}\sin 6\theta + \cdots}{1 + n\cos 2\theta + \frac{n(n-1)}{2!}\cos 4\theta + \frac{n(n-1)(n-2)}{3!}\cos 6\theta +}.$$

<u>Exercise:</u> If possible, find similar identities to the above using $\left(1-e^{2i\theta}\right)^n$, $\left(1+e^{i\theta}\right)^n$, and $\left(1-e^{i\theta}\right)^n$.

5) Suppose we want to show that, for any real numbers p and m,

$$e^{2mi\cot^{-1}p}\left(\frac{pi+1}{pi-1}\right)^m = 1\,.$$

How do we start? We could take the m^{th} roots on both sides, and then work $1^{1/m}$. Or we could try to convert $e^{2i.\cot^{-1}p}$ in Cartesian form and then combine it with $(pi+1)/(pi-1)$, and then take the m^{th} roots. However, given the power and ease of use of the exponential form of a complex number, it makes sense to try to convert $(pi+1)/(pi-1)$ into expoenential form. In fact, our first thought when working with complicated complex numbers problems should be to convert everything into exponential form. For more simple problems you can also consider converting a complex number into polar form, if you wish. Therefore,

- let $z_1 = 1 + pi$. Then $r_1 = |z_1|$. For $\arg(z_1)$ we need to consider the cases of $p \geq 0$ and $p < 0$, so as to take the correct argument. In both cases we have $\theta_1 = \arg(z_1) = \tan^{-1}p$;
- let $z_2 = -1 + pi$. Then $r_2 = |z_2|$. As above, for $\arg(z_2)$ we need to consider the cases of $p \geq 0$ and $p < 0$, so as to take the correct argument. In both cases we have $\theta_2 = \arg(z_2) = \tan^{-1}(-p) - \pi = -\tan^{-1}(p) - \pi$;
- Notice that $r_1 = r_2$, and also that there is no need to evaluated these since they will cancel out by the division;

Hence we have

$$
\begin{aligned}
e^{2mi\cot^{-1}p}\left(\frac{pi+1}{pi-1}\right)^m &= e^{2mi\cot^{-1}p}\left(\frac{e^{i\tan^{-1}p}}{e^{-i(\tan^{-1}p-\pi)}}\right)^m , \\
&= e^{2mi\cot^{-1}p}\left(e^{i\tan^{-1}p}.\, e^{i(\tan^{-1}p-\pi)}\right)^m , \\
&= e^{2mi\cot^{-1}p}\left(e^{2i\tan^{-1}p-i\pi}\right)^m , \\
&= e^{2mi\cot^{-1}p}.\, e^{2mi\tan^{-1}p-mi\pi} , \\
&= e^{2mi(\cot^{-1}p+\tan^{-1}p)}.\, e^{-mi\pi} .
\end{aligned}
$$

By standard inverse trig work (left as an exercise), $\cot^{-1}p+\tan^{-1}p=\pi/2$, for all real values of p. Therefore

$$e^{2mi(\cot^{-1}p+\tan^{-1}p)}e^{-mi\pi}=e^{2mi\pi/2}e^{-mi\pi}=e^0=1 ,$$

this being true for all values of m. hence we have shown that

$$e^{2mi\cot^{-1}p}\left(\frac{pi+1}{pi-1}\right)^m=1$$

for all real values of p and m.

1.19 Properties of $|z|$ and $\arg(z)$

1.19.1 Properties of $|z|$

We will start with some simple proofs in order to illustrate the nature of presenting proofs. Once we have got used to this we can move onto proving more complicated statements.

<u>Property 1:</u> Given $z=x+iy$ prove $|-z|=|z|$.

Proof: Since $z=x+iy$ we have $-z=-x-iy$. Then $|z|=\sqrt{x^2+y^2}$ and $|-z|=\sqrt{(-x)^2+(-y)^2}=\sqrt{x^2+y^2}$. Hence $|-z|=|z|$. ■

Comment: The nature of a proof is that every mathematical statement beyond what is already given, or known prior, has to be explcitely developed. So, since $-z$ is not given to us we must develop the step which leads to $-z$. Now, this may seem trivial, but it is in the nature of proofs that we should do this explicitly, hence the reason for me writing "$-z=-x-iy$". More than this, I have presented this step in such a way as to read as a direct consequence of $z=x+iy$.

The same reasoning applies to justify writing down the expression for $|z|$ and $|-z|$, with the added specificity of presenting the latter as $\sqrt{(-x)^2 + (-y)^2}$ and not as $\sqrt{x^2 + y^2}$. This is to explicitly illustrate the effect of the mathematical operations on the negative components, and that such operations simplify in such a way as to be equal to the positive components. We then state this equality explicitely as the last statement of the proof.

<u>Property 2:</u> Given $z = x + iy$ it is left as an exercise to prove $|z| = |z^*|$.

<u>Property 3:</u> Given $z = x + iy$ prove $z^*z = |z|^2$.

Proof: Since $z = x + iy$ we have $z^* = x - iy$. Therefore $z^*z = (x + iy)(x - iy) = x^2 + y^2$. But $|z|^2 = \left(\sqrt{x^2 + y^2}\right)^2 = x^2 + y^2$. Hence $z^*z = |z|^2$. ■

<u>Property 4:</u> Given two complex numbers z and w prove $|z.w| = |z||w|$.

Proof: Let $z = x + iy$ and $w = u + iv$. Then $|z| = \sqrt{x^2 + y^2}$ and $|w| = \sqrt{u^2 + v^2}$. Hence

$$|z||w| = \sqrt{x^2 + y^2}.\sqrt{u^2 + v^2} = \sqrt{(x^2 + y^2)(u^2 + v^2)}\,.$$

Separately we have $z.w = (xu - yv) + i(xv + yu)$, hence $|z.w| = \sqrt{(xu - yv)^2 + (xv + yu)^2}$. Expanding and simplifying this gives

$$|z.w| = \sqrt{x^2u^2 + y^2v^2 + x^2v^2 + y^2u^2} = \sqrt{(x^2 + y^2)(u^2 + v^2)}\,.$$

Therefore $|z.w| = |z||w|$. ■

<u>Property 5:</u> Given two complex numbers z and w it is left as an exercise to prove $|z/w| = |z|/|w|$.

<u>Property 6:</u> Given a complex numbers z prove $z^* = 1/z$ if and only if $|z| = 1$.

Proof: Let $z = x + iy$. Then

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{1}{x + iy}.\frac{x - iy}{x - iy} = \frac{x - iy}{x^2 + y^2}\,.$$

If $|z| = 1$ we have $|z| = \sqrt{x^2 + y^2} = 1$ implies $x^2 + y^2 = 1$. Hence

$$\frac{1}{z} = x - iy = z^*$$

as required.

On the other hand we can form the modulus of $z^* = 1/z$ thus giving $|z^*| = |1/z|$. By property 2 we have $|z^*| = |z|$, hence we have $|z| = |1/z|$. By property 5 this becomes $|z| = 1/|z|$. Cross-multiplying by $|z|$ we obtain $|z|^2 = 1$ which implies $|z| = 1$, as required. ∎

Comment: Here we have had to do two version sof the proof. Thje reason for this is in the wording "if and only if". What this means is that i) given $|z| = 1$ then $z^* = 1/z$, and ii) given $z^* = 1/z$ then $|z| = 1$. So we have to prove each statement leads to the other statement.

Exercises:

1) Given $z = x + iy$ prove $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$
2) Is it true that, for two complex number z and w, $|z^* + w| = |z + w^*|$. If so, prove it, otherwise find a counterexample.

1.19.2 Properties of $\arg(z)$

Having gained some experience of using DeMoivre's theorem we can now prove certain properties of the arguments of a complex number. There are two points which are worth highlighting when reading the proofs below:

- We must remember that the principal argument of a complex number z is denoted $\theta = \mathrm{Arg}(z)$ where $-\pi < \mathrm{Arg}(z) \leq \pi$.
- The key in all of the proofs below is the way in which the periodicity $2k\pi$ is taken into account as part of the argument of a complex number, and the way in which this is analysed in order to make sure that our argument lies in $(-\pi, \pi]$. As such, in nearly all the proofs below we first expand and simplify the given complex number, and then we take account of the periodicity of cos and sin of the simplified complex number. Finally we find the relevant values of k which bring the argument of this simplified complex number back into the interval $(-\pi, \pi]$.

<u>Property 1:</u> $\mathrm{Arg}(z^*) = -\mathrm{Arg}(z) + 2k\pi$, where $k = 0, 1$.

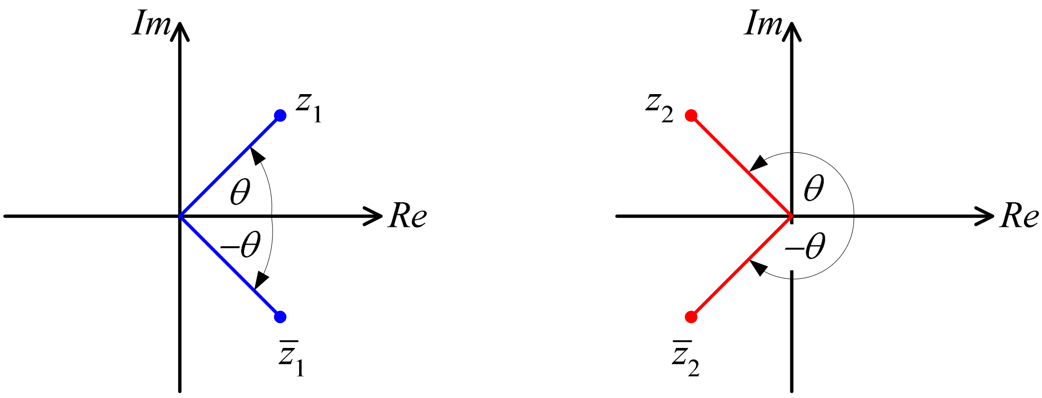
Proof: Let $z = r(\cos\theta + i\sin\theta)$. Then

$$\begin{aligned} z^* &= r(\cos\theta - i\sin\theta)\,, \\ &= r(\cos(-\theta) + i\sin(-\theta))\,, \\ &= r(\cos(-\theta + 2k\pi) + i\sin(-\theta + 2k\pi))\,. \end{aligned}$$

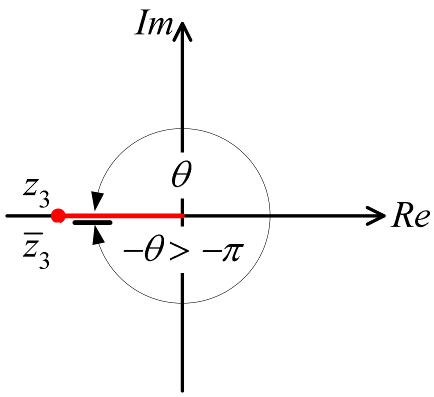
By definition, $\theta \in (-\pi, \pi]$ and also $-\theta \in (-\pi, \pi]$. Hence $\operatorname{Arg}(z^*) = -\operatorname{Arg}(z)$. However, $\theta = \pi$ implies $-\theta = -\pi \notin (-\pi, \pi]$. But $-\theta + 2\pi = \pi \in (-\pi, \pi]$, hence $\operatorname{Arg}(z^*) = -\operatorname{Arg}(z) + 2\pi$ when $\theta = \pi$. Therefore

$$\operatorname{Arg}(z^*) = -\operatorname{Arg}(z) + 2k\pi, \text{ where } k = 0, 1. \qquad \blacksquare$$

Comment: Property 1 can be illustrated as shown below



Both θ and $-\theta$ lie in the interval $(-\pi, \pi]$.



θ lies in the interval $(-\pi, \pi]$ but $-\theta = -\pi$ does not.

<u>Property 2:</u> $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) + 2k\pi$, for $k = -1, 0, +1$.

Proof: Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_2)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, where $\theta_1 \in (-\pi, \pi]$ and $\theta_2 \in (-\pi, \pi]$. Then we have $\operatorname{Arg}(z_1) = \theta_1$ and $\operatorname{Arg}(z_2) = \theta_2$.

Then,

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\,, \\ &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))\,, \\ &= r_1 r_2(\cos(\theta_1 + \theta_2 + 2k\pi) + i\sin(\theta_1 + \theta_2 + 2k\pi))\,. \end{aligned}$$

Hence

$$\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) = \theta_1 + \theta_2 + 2k\pi$$

for some value $k \in \mathbb{Z}$.

Given that $\theta_1 \in (-\pi, \pi]$ and $\theta_2 \in (-\pi, \pi]$ we have $\theta_1 + \theta_2 \in (-2\pi, 2\pi]$. However, we want $\theta_1 + \theta_2 \in (-\pi, \pi]$. So

- for $\theta_1 + \theta_2 \in (-2\pi, -\pi]$ we have $\theta_1 + \theta_2 + 2k\pi \in (-2\pi + 2k\pi, -\pi + 2k\pi]$. When $k = 1$ we obtain $\theta_1 + \theta_2 + 2\pi \in (0, \pi]$ which is within $(-\pi, \pi]$. Hence

$$\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) = \theta_1 + \theta_2 + 2\pi\ ;$$

- for $\theta_1 + \theta_2 \in (-\pi, \pi]$ we have $\theta_1 + \theta_2 + 2k\pi \in (-\pi + 2k\pi, \pi + 2k\pi]$. When $k = 0$ we obtain $\theta_1 + \theta_2 \in (-\pi, \pi]$ which is our principal interval. Hence

$$\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) = \theta_1 + \theta_2\ ;$$

- for $\theta_1 + \theta_2 \in (\pi, 2\,\pi]$ we have we have $\theta_1 + \theta_2 + 2k\pi \in (\pi + 2k\pi, 2\pi + 2k\pi]$. When $k = -1$ we obtain $\theta_1 + \theta_2 - 2\pi \in (-\pi, 0]$ which is within $(-\pi, \pi]$. Hence

$$\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) = \theta_1 + \theta_2 - 2\pi\ .$$

Therefore $\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) + 2k\pi$, for $k = -1, 0, +1$, depending upon whether $\theta_1 + \theta_2$ is greater than π, lies in $(-\pi, \pi]$, or is less than $-\pi$ respectively. ■

Comment 1: You might have thought that $\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$. This is a mistake as we see from the proof above. It is an easy mistake to make, and highlights the fact that we need to be vigilant when it comes to dealing arguments of complex numbers. The reason is simply due to the periodic nature of cos and sin, and this must be taken into account at the appropriate stage in our proofs or calculations.

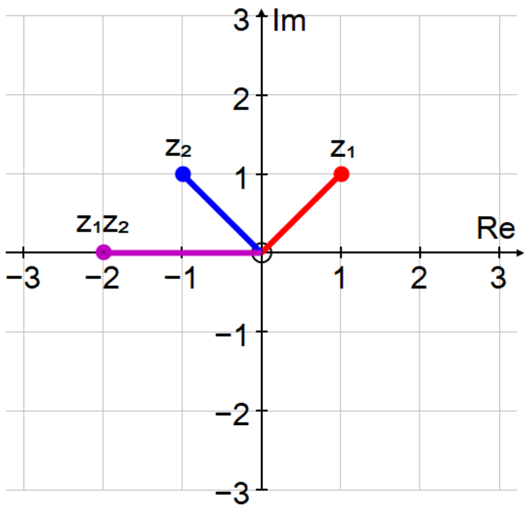
Comment 2: Since it is generally likely that $\theta_1 + \theta_2$ will lie outside the interval $(-\pi, \pi]$ we need to find out what "correction" needs to be made to bring $\theta_1 + \theta_2$ back into this interval. Hence the need to study subintervals of $(-2\pi, 2\pi]$. These subintervals can be done "2π at a time", in other words $\ldots, (-3\pi, -\pi], (-\pi, \pi], (\pi, 3\pi], \ldots$.

Example 1:

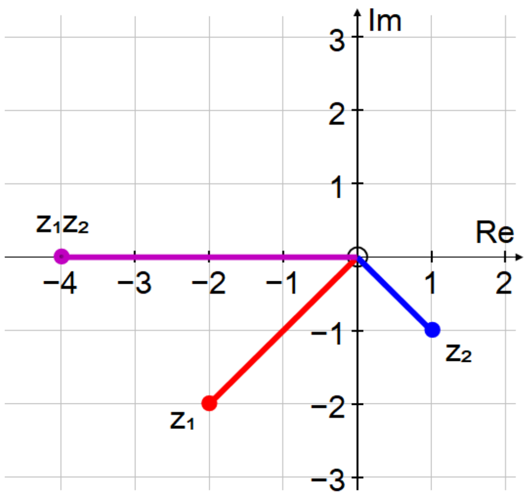
Let $z_1 = 1 + i$ and $z_2 = -1 + i$. Then $\mathrm{Arg}(z_1) = \pi/4$ and $\mathrm{Arg}(z_2) = 3\pi/4$, therefore $\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) = \pi$. Also, $z_1 z_2 = -5$ therefore $\mathrm{Arg}(z_1 z_2) = \pi$. Hence $\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$.

Example 2: Let $z_1 = -2 - 2i$ and $z_2 = 1 - i$. Then $\text{Arg}(z_1) = -3\pi/4$ and $\text{Arg}(z_2) = -\pi/4$, therefore $\text{Arg}(z_1) + \text{Arg}(z_2) = -\pi$. But $z_1 z_2 = -4$ implying $\text{Arg}(z_1 z_2) = \pi$. Therefore $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) + 2\pi$.

Below are graphs representing the complex numbers for the above two examples.



Graph for example 1

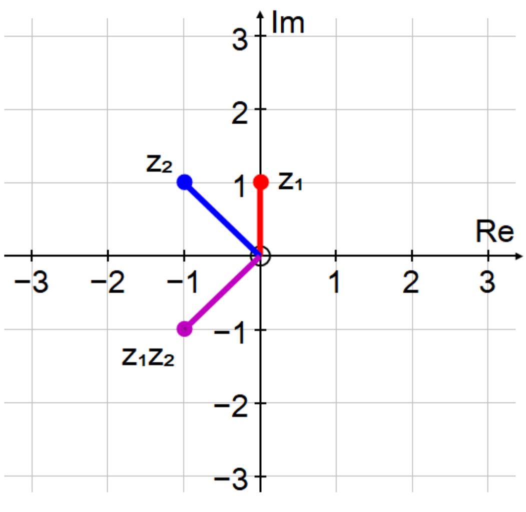


Graph for example 2

Example 3:

Let $z_1 = i$ and $z_2 = -1 + i$. Then $\text{Arg}(z_1) = \pi/2$ and $\text{Arg}(z_2) = 3\pi/4$, therefore $\text{Arg}(z_1) + \text{Arg}(z_2) = 5\pi/4$. But $z_1 z_2 = -1 - i$ implying $\text{Arg}(z_1 z_2) = -3\pi/4$. Therefore $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) - 2\pi$.

Below are graphs representing the complex numbers for the above example



Graph for example 3

Exercise:

Consider $z = i$ and $z = -1 + i$. For both complex numbers find $\operatorname{Arg}(z^2)$ and $\operatorname{Arg}(z) + \operatorname{Arg}(z)$. What can you say about $\operatorname{Arg}(z) + \operatorname{Arg}(z)$ and $2\operatorname{Arg}(z)$? What is the correct result for $\operatorname{Arg}(z) + \operatorname{Arg}(z)$? What is the relationship between $\operatorname{Arg}(z^2)$ and $\operatorname{Arg}(z) + \operatorname{Arg}(z)$. Prove this and state a similar result for $\operatorname{Arg}(z^n)$.

Property 3: $\operatorname{Arg}(z_1/z_2) = \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2) + 2k\pi$, where $k = -1, 0, +1$.

Proof: This is left as an exercise. The proof follows exactly the same logic as that shown for the proof of property 2. ■

Property 4: $\operatorname{Arg}(1/z) = -\operatorname{Arg}(z)$.

Proof: Let $z = r(\cos\theta + i\sin\theta)$. Then $\theta = \operatorname{Arg}(z)$. Hence, $1/z = z^{-1} = r^{-1}(\cos\theta + i\sin\theta)^{-1}$ implying

$$z^{-1} = r^{-1}(\cos(-\theta) + i\sin(-\theta)) = r^{-1}(\cos(-\theta + 2k\pi) + i\sin(-\theta + 2k\pi)) .$$

Therefore $\operatorname{Arg}(1/z) = -\theta + 2k\pi$. Since $-\theta \in (-\pi, \pi]$, k has to be zero. Hence $\operatorname{Arg}(1/z) = -\theta$, i.e. $\operatorname{Arg}(1/z) = -\operatorname{Arg}(z)$. ■

Property 5: $\operatorname{Arg}(z.\bar{z}_2) = \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2) + 2k\pi$, for $k = -1, 0, +1$

Proof: This is left as an exercise. It follows the same logic as property 2. ■

Property 6: $\operatorname{Arg}(z/z^*) = 2\operatorname{Arg}(z) + 2k\pi$, for $k = -1, 0, +1$

Proof: This is left as an exercise. ■

Property 7: $\operatorname{Arg}(z^n) = n\operatorname{Arg}(z) + 2k\pi$, for $k = -1, 0, +1$

Proof: This is left as an exercise. ■

Property 8: $\operatorname{Arg}(z_2/z_1) = 2k\pi - \operatorname{Arg}(z_1/z_2)$, for $k = -1, 0, +1$

Proof: Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, where $\theta_1 \in (-\pi, \pi]$ and $\theta_2 \in (-\pi, \pi]$. Then we have $\operatorname{Arg}(z_1) = \theta_1$ and $\operatorname{Arg}(z_2) = \theta_2$.

Then,

$$\begin{aligned}
\frac{z_2}{z_1} &= \frac{r_2}{r_1}\frac{\cos\theta_2 + i\sin\theta_2}{\cos\theta_1 + i\sin\theta_1}, \\
&= \frac{r_2}{r_1}(\cos\theta_2 + i\sin\theta_2)(\cos\theta_1 + i\sin\theta_1)^{-1}, \\
&= \frac{r_2}{r_1}(\cos\theta_2 + i\sin\theta_2)(\cos(-\theta_1) + i\sin(-\theta_1)), \\
&= \frac{r_2}{r_1}(\cos(\theta_2 - \theta_1) + i\sin(\theta_2 - \theta_1)), \\
&= \frac{r_2}{r_1}(\cos(\theta_2 - \theta_1 + 2k\pi) + i\sin(\theta_2 - \theta_1 + 2k\pi)).
\end{aligned}$$

Hence

$$\operatorname{Arg}\left(\frac{z_2}{z_1}\right) = \theta_2 - \theta_1 + 2k\pi$$

for some value $k \in \mathbb{Z}$.

Given that $\theta_1 \in (-\pi, \pi]$ and $\theta_2 \in (-\pi, \pi]$ we have $\theta_2 - \theta_1 \in (-2\pi, 2\pi]$. However, we want $\theta_2 - \theta_1 \in (-\pi, \pi]$. So

- for $\theta_2 - \theta_1 \in (-2\pi, -\pi]$ we have $\theta_2 - \theta_1 + 2k\pi \in (-2\pi + 2k\pi, -\pi + 2k\pi]$. When $k = 1$ we obtain $\theta_2 - \theta_1 + 2\pi \in (0, \pi]$ which is within $(-\pi, \pi]$. Therefore $-(\theta_1 - \theta_2) + 2\pi \in (0, \pi]$. Hence

$$\operatorname{Arg}\left(\frac{z_2}{z_1}\right) = \theta_2 - \theta_1 + 2\pi = 2\pi - (\theta_1 - \theta_2);$$

- for $\theta_2 - \theta_1 \in (-\pi, \pi]$ we have $\theta_2 - \theta_1 + 2k\pi \in (-\pi + 2k\pi, \pi + 2k\pi]$. When $k = 0$ we obtain $\theta_2 - \theta_1 \in (-\pi, \pi]$ which is our principal interval. Therefore $-(\theta_1 - \theta_2) \in (0, \pi]$. Hence

$$\operatorname{Arg}\left(\frac{z_2}{z_1}\right) = \theta_2 - \theta_1 = -(\theta_1 - \theta_2);$$

- for $\theta_2 - \theta_1 \in (\pi, 2\,\pi]$ we have we have $\theta_2 - \theta_1 + 2k\pi \in (\pi + 2k\pi, 2\pi + 2k\pi]$. When $k = -1$ we obtain $\theta_2 - \theta_1 - 2\pi \in (-\pi, 0]$ which is within $(-\pi, \pi]$. Therefore $-(\theta_1 - \theta_2) - 2\pi \in (-\pi, \pi]$Hence

$$\operatorname{Arg}\left(\frac{z_2}{z_1}\right) = \theta_2 - \theta_1 - 2\pi = -2\pi - (\theta_1 - \theta_2);$$

Hence $\operatorname{Arg}(z_2/z_1) = 2k\pi - \operatorname{Arg}(z_1/z_2)$, for $k = -1, 0, +1$ ■

Exercise: Prove $\arg(z^*) = 2k\pi - \arg(z)$, and find the range of values of k for which this is valid.

Property 9: $\operatorname{Arg}(z - z^*) = \pm\pi/2$, where z is not pure real.

Proof: Let $z = x + iy$. Then $z^* = x - iy$. Therefore, $z - z^* = x + iy - (x - iy) = 2iy$. Hence $\operatorname{Arg}(z - z^*) = \tan^{-1}(2y/0) = \pm\pi/2$. If $z = x$ then $z - z^* = 0$, therefore $\operatorname{Arg}(z - z^*) = \tan^{-1}(0/0)$ which is undefined. Hene $\operatorname{Arg}(z - z^*) = \pm\pi/2$, where z is not pure real. ■

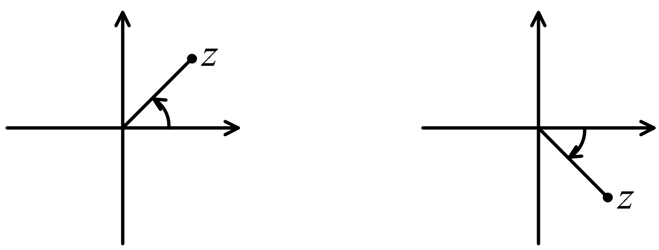
Exercise: Find a similar result for $\operatorname{Arg}(z + z^*)$ and prove it. What constraints, if any, are there on z?

Property 10: $\operatorname{Arg}(-z) = \operatorname{Arg}(z) + k\pi$, where $k = \pm 1$

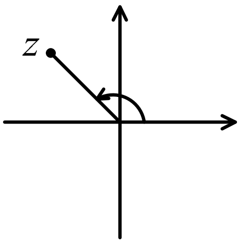
Proof: We proceed on a case by case basis by considering the argument of z and $-z$ for $x \lessgtr 0$ and $y \lessgtr 0$. Hence

- for $z = x + iy$ we have
  - i) $\operatorname{Arg}(z) = \tan^{-1}(y/x)$ for $x > 0, y > 0$ and $x > 0, y < 0$ illustrated as diagrams (a) and (b) below.
  - ii) $\operatorname{Arg}(z) = \tan^{-1}(y/x) \pm \pi$ for $x < 0, y > 0$ and for $x < 0, y < 0$, illustrated as diagrams (c) and (d) below.

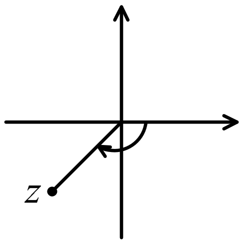
- for $-z = -x - iy$ we have
  - i) $\operatorname{Arg}(-z) = \tan^{-1}(y/x)$ for $x < 0, y < 0$ and $x < 0, y > 0$ illustrated as diagrams (a) and (b) below.
  - ii) $\operatorname{Arg}(-z) = \tan^{-1}(y/x) \pm \pi$ for $x > 0, y < 0$ and $x > 0, y > 0$ illustrated as diagrams (c) and (d) below.



(a) (b)



(c)



(d)

So in general we have $\text{Arg}(-z) = \text{Arg}(z) + k\pi$, where $k = \pm 1$ ■

Exercises: Prove i) $\text{Arg}(z.z^*) = 0$, ii) $\text{Arg}(z + z^*) = 0$.

<u>Property 11:</u> $\text{Arg}\big((z_3 - z_2)/(z_3 - z_1)\big) = \frac{1}{2}\text{Arg}(z_1/z_2)$ for three distinct complex numbers z_1, z_2, z_3

Proof: Since the argument of a complex number remains unaffected by its modulus/length we let $z_1 = \cos\theta_1 + i\sin\theta_1$, $z_2 = \cos\theta_2 + i\sin\theta_2$, and $z_3 = \cos\theta_3 + i\sin\theta_3$ without any loss of generality. Therefore, $\text{Arg}(z_1) = \theta_1$, $\text{Arg}(z_2) = \theta_2$, $\text{Arg}(z_3) = \theta_3$.

Also,

$$\begin{aligned} z_3 - z_2 &= \cos\theta_3 + i\sin\theta_3 - \cos\theta_2 - i\sin\theta_2\,, \\ &= \cos\theta_3 - \cos\theta_2 + i(\sin\theta_3 - \sin\theta_2)\,. \end{aligned}$$

Using the factor formula from the trig family of identities we obtain

$$\begin{aligned} z_3 - z_2 &= -2\sin\left(\frac{\theta_3+\theta_2}{2}\right)\sin\left(\frac{\theta_3-\theta_2}{2}\right) + i\left(2\sin\left(\frac{\theta_3-\theta_2}{2}\right)\cos\left(\frac{\theta_3+\theta_2}{2}\right)\right), \\ &= 2\sin\left(\frac{\theta_3-\theta_2}{2}\right)\left[\cos\left(\frac{\theta_3+\theta_2}{2}\right) - i\sin\left(\frac{\theta_3+\theta_2}{2}\right)\right], \\ &= 2\sin\left(\frac{\theta_3-\theta_2}{2}\right)\left[\cos\left(-\frac{\theta_3+\theta_2}{2}\right) + i\sin\left(-\frac{\theta_3+\theta_2}{2}\right)\right]. \end{aligned}$$

Similarly

$$z_3 - z_1 = 2\sin\left(\frac{\theta_3-\theta_1}{2}\right)\left[\cos\left(-\frac{\theta_3+\theta_1}{2}\right) + i\sin\left(-\frac{\theta_3+\theta_1}{2}\right)\right].$$

Hence

$$\frac{z_3 - z_2}{z_3 - z_1} = a\frac{\cos\left(-\frac{\theta_3 + \theta_2}{2}\right) + i\sin\left(-\frac{\theta_3 + \theta_2}{2}\right)}{\cos\left(-\frac{\theta_3 + \theta_1}{2}\right) + i\sin\left(-\frac{\theta_3 + \theta_1}{2}\right)}$$

where $a = \sin\left(\frac{\theta_3 - \theta_2}{2}\right) / \sin\left(\frac{\theta_3 - \theta_1}{2}\right)$. Therefore

$$\begin{aligned}
\frac{z_3 - z_2}{z_3 - z_1} &= a\left(\cos\left(-\frac{\theta_3 + \theta_2}{2}\right) + i\sin\left(-\frac{\theta_3 + \theta_2}{2}\right)\right)\left(\cos\left(-\frac{\theta_3 + \theta_1}{2}\right) + i\sin\left(-\frac{\theta_3 + \theta_1}{2}\right)\right)^{-1}, \\
&= a\left(\cos\left(-\frac{\theta_3 + \theta_2}{2}\right) + i\sin\left(-\frac{\theta_3 + \theta_2}{2}\right)\right)\left(\cos\left(\frac{\theta_3 + \theta_1}{2}\right) + i\sin\left(\frac{\theta_3 + \theta_1}{2}\right)\right), \\
&= a\left(\cos\left(-\frac{\theta_3}{2} - \frac{\theta_2}{2} + \frac{\theta_3}{2} + \frac{\theta_1}{2}\right) + i\sin\left(-\frac{\theta_3}{2} - \frac{\theta_2}{2} + \frac{\theta_3}{2} + \frac{\theta_1}{2}\right)\right), \\
&= a\left(\cos\left(\frac{\theta_1}{2} - \frac{\theta_2}{2}\right) + i\sin\left(\frac{\theta_1}{2} - \frac{\theta_2}{2}\right)\right), \\
&= a\left(\cos\left(\frac{\theta_1}{2} - \frac{\theta_2}{2} + 2k\pi\right) + i\sin\left(\frac{\theta_1}{2} - \frac{\theta_2}{2} + 2k\pi\right)\right).
\end{aligned}$$

Hence

$$\operatorname{Arg}\big((z_3 - z_2)/(z_3 - z_1)\big) = \frac{\theta_1}{2} - \frac{\theta_2}{2} + 2k\pi\,,$$

for some $k \in \mathbb{Z}$. Since $\theta_1, \theta_2 \in (-\pi, \pi]$, $\theta_1 - \theta_2 \in (-2\pi, 2\pi]$ implying $(\theta_1 - \theta_2)/2 \in (-\pi, \pi]$. Therefore $k = 0$ and we have $\operatorname{Arg}\big((z_3 - z_2)/(z_3 - z_1)\big) = (\theta_1 - \theta_2)/2$, i.e.

$$\operatorname{Arg}\big((z_3 - z_2)/(z_3 - z_1)\big) = \frac{1}{2}(\operatorname{Arg}(z_1) - \operatorname{Arg}(z_2)) = \frac{1}{2}\operatorname{Arg}\left(\frac{z_1}{z_2}\right).$$

■